Fundamentals of Borehole Logging AES1500, 8 April 2005, 1400-1700

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1) A sponge and an oil reservoir are two examples of a poroelastic material, comprising solid material and fluid-filled pores. Mass conservation for the solid and the fluid gives:

$$\frac{\partial}{\partial t}(1-\phi)\rho_s + \frac{\partial}{\partial x}(1-\phi)\rho_s v = 0, \tag{1}$$

$$\frac{\partial}{\partial t}\phi\rho_f + \frac{\partial}{\partial x}\phi\rho_f w = 0 \tag{2}$$

where ϕ, v, and w are the porosity, the velocity of the solid, and the velocity of the fluid. The solid and the fluid densities are denoted ρ_s and ρ_f, respectively.

a) Linearize (1) and (2) by substitution of $\rho_s = \rho_{s0} + \rho_s'$, $\rho_f = \rho_{f0} + \rho_f'$, $v = v'$, $w = w'$, $\phi = \phi_0 + \phi'$.

b) Under what condition can (1) can be rewritten as

$$\frac{\partial \phi'}{\partial t} = (1-\phi_0)\frac{\partial v'}{\partial x}? \tag{3}$$

c) Write the storage equation, which is obtained from substitution of (3) into the linearized version of (2).

Hooke's law for poro-elastic material can be written as follows:

$$-\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + \frac{K_b}{K_s}p\delta_{ij}, \tag{4}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{5}$$

where K_b and K_s are the bulk moduli (incompressibilities) of the matrix and the individual grains, respectively, and p is the pore pressure.

d) Use (4) and (5) to elaborate σ_{11}. What is the meaning of σ_{11}? What is the meaning of u_i and u_j?

e) What becomes of your expression for σ_{11} if the grains are incompressible?

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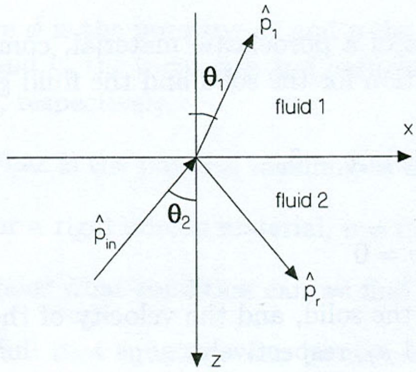
2) A plane monochromatic wave is incident from below upon the interface between fluids 1 and 2 (see figure).



a) What is the formula to compute the wave velocities in fluids 1 and 2?

b) In the figure, $\theta_1 < \theta_2$. Is this correct if fluids 1 and 2 have identical compressibilities?

The pressure in fluid 1 is given by:

$$p_1 = \hat{p}_1 \exp i(\omega t - k_x x + k_{1z} z) \qquad (6)$$

c) Give the corresponding expressions for p_{in} and p_r.

Momentum conservation yields that

$$\rho_1 \frac{\partial w_1}{\partial t} = -\frac{\partial p_1}{\partial z} \qquad (7)$$

d) What is w_1 and what is its unit?

Now write

$$w_1 = \hat{w}_1 \exp i(\omega t - k_x x + k_{1z} z) \qquad (8)$$

e) Substitute (6) and (8) into (7), and give the relation between $\hat{w}_1$ en $\hat{p}_1$, using the impedance definition $Z_1 = \omega\rho_1/k_{1z} = \omega\rho_1/(k_1 \cos\theta_1)$.

f) Give the corresponding expressions for the relations between $\hat{w}_{in}$ and $\hat{p}_{in}$, and between $\hat{w}_r$ en $\hat{p}_r$.

g) Give the boundary conditions for pressure and velocity at the interface between the two fluids and derive an expression for the reflection coefficient $R = \hat{p}_r/\hat{p}_{in}$. In this expression the angles θ_1 and θ_2 should appear.

h) Snell's law tells us that $\sin(\theta_1)/c_1 = \sin(\theta_2)/c_2$, where c_1 and c_2 are the wave velocities in the fluids 1 en 2. For $c_1/c_2 = \sqrt{2}$ and $\theta_2 = 30^o$, compute θ_1.

i) Compute the reflection coefficient for $\rho_2/\rho_1 = 2$ (light fluid on top).

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3) Along the interface between a solid and a vacuum, Rayleigh surface waves may propagate.

a) How do we call the surface wave in case the vacuum is replaced by a liquid? What is a so-called tube wave?

The Helmholtz decomposition for the displacements of a solid is defined by:

$$\underline{u} = \nabla\Phi + \nabla \times \underline{\Psi} \tag{9}$$

b) Write the expressions for the three components u_1, u_2, and u_3. Note that $\underline{\Psi}$ is a vector, having components Ψ_1, Ψ_2, and Ψ_3 in the x, y, and z directions respectively.

For 2D wave propagation in the XOZ plane, we know that $u_2 = 0$, and that $\partial/\partial y = 0$. So we now know that

$$\underline{u} = \begin{pmatrix} \dfrac{\partial\Phi}{\partial x} - \dfrac{\partial\Psi}{\partial z} \\ 0 \\ \dfrac{\partial\Phi}{\partial z} + \dfrac{\partial\Psi}{\partial x} \end{pmatrix}, \tag{10}$$

where Ψ is a shorthand notation for Ψ_2. Hooke's law gives the relation between stresses and displacements:

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \tag{11}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{12}$$

c) Write σ_{31} and σ_{33} as a function of the potentials Φ en Ψ.

For a plane monochromatic wave, we may write

$$\Phi = \hat{\Phi} \exp i(\omega t - k_x x - k_{1z} z) \tag{13}$$

$$\Psi = \hat{\Psi} \exp i(\omega t - k_x x - k_{3z} z) \tag{14}$$

d) What is the meaning of k_{1z} en k_{3z} ?

e) Now rewrite σ_{31} and σ_{33}

f) On the interface $z = 0$ between an elastic medium and a vacuum, it holds that $\sigma_{31} = \sigma_{33} = 0$. Both boundary conditions are met by a Rayleigh wave. Derive the dispersion relation for this wave (Hint: write both boundary conditions in terms of the matrix relation $\underline{\underline{A}}\,\underline{x} = 0$, and calculate the determinant of matrix $\underline{\underline{A}}$).

4) The Biot momentum equation for the fluid in an elastic porous material is

$$\phi\rho_f\frac{\partial w}{\partial t}=-\phi\frac{\partial p}{\partial x}+\frac{\eta\phi^2}{k_0}(v-w)+(\alpha_\infty-1)\phi\rho_f\frac{\partial}{\partial t}(v-w), \tag{15}$$

where ϕ is the porosity, ρ_f and η the fluid density and viscosity, p the fluid pore pressure, α_∞ and k_0 the tortuosity and permeability, and w and v the velocities of the fluid and the solid, respectively.

a) What is the physical meaning of the tortuosity ?

b) For a rigid porous material, $v = 0$. Now rewrite (15) until only three terms are left.

c) Under what condition can we find back Darcy's law ? Write Darcy's law.

The full Biot equations can be cast in the following form:

$$\begin{aligned}\hat{w}[-R+c^2\hat{\rho}_{22}] &= \hat{v}[\;Q-c^2\hat{\rho}_{12}]\\ \hat{w}[-Q+c^2\hat{\rho}_{12}] &= \hat{v}[\;P-c^2\hat{\rho}_{11}],\end{aligned} \tag{16}$$

where we have defined the density terms:

$$\begin{aligned}\hat{\rho}_{12} &= -(\alpha_\infty-1)\phi\rho_f+ib/\omega\\ \hat{\rho}_{11} &= (1-\phi)\rho_s-\hat{\rho}_{12}\\ \hat{\rho}_{22} &= \phi\rho_f-\hat{\rho}_{12}\end{aligned} \tag{17}$$

and the incompressibilities

$$\begin{aligned}P &= K_p+K_f\frac{(1-\phi)^2}{\phi}\\ Q &= (1-\phi)K_f\\ R &= \phi K_f\end{aligned} \tag{18}$$

d) Show that set (16) yields the *dispersion relation* for the unknown wave velocity c:

$$(\hat{\rho}_{11}\hat{\rho}_{22}-\hat{\rho}_{12}^2)c^4-\Delta c^2+(PR-Q^2)=0, \tag{19}$$

and give the expression for Δ.

e) Solve (19).