‘Advanced reflection seismology and seismic imaging’ (AES 1560-1)

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1a. Consider the law of mass conservation

$$\frac{\partial}{\partial t}\int_{\mathbb{V}} \rho\, d\mathbb{V} = -\oint_{\mathbb{S}} \rho\vec{v}\cdot\vec{n}\, d\mathbb{S} + \frac{\partial}{\partial t}\int_{\mathbb{V}} i_m\, d\mathbb{V}.$$

Use the theorem of Gauss to change the surface integral into a volume integral. Derive the non-linear equation of continuity.

1b. Linearize the equation of continuity for a non-flowing medium, using $\rho(\vec{r},t) = \rho_0(\vec{r}) + \Delta\rho(\vec{r},t)$, assuming $\Delta\rho(\vec{r},t) << \rho_0(\vec{r})$ and $|\nabla\rho_0(\vec{r})| << |\nabla(\Delta\rho(\vec{r},t))|$. Substitute the linearized equation of state $\frac{\Delta\rho}{\rho_0} = \frac{1}{K}\Delta p$.

1c. Linearize the non-linear equation of motion $\frac{\partial(\rho\vec{v})}{\partial t} + \vec{v}\nabla\cdot(\rho\vec{v}) + (\rho\vec{v}\cdot\nabla)\vec{v} + \nabla p = \vec{f}$ in a similar way. Rewrite the linearized equations of continuity and motion by making the substitutions $\Delta p(\vec{r},t) \to p(\vec{r},t)$ and $\rho_0(\vec{r}) \to \rho(\vec{r})$. Derive the wave equation for an inhomogeneous medium by eliminating $\vec{v}$.

1d. For a homogeneous source-free medium, the result of 1c reduces to $\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$, where c is the propagation velocity. Derive an expression for c in terms of K and ρ.

1e. Show that the plane wave expression $p(x,y,z,t) = u(t - s_x x - s_y y - s_z z)$ is a solution of the wave equation in 1d and derive a relation between the slownesses s_x, s_y, s_z and the propagation velocity c.

1f. Use the linearized equation of motion to show that the particle velocity of the plane wave in 1e is parallel to the propagation direction of this plane wave.

1g. Consider a two-dimensional monochromatic homogeneous plane wave $p(x,z,t) = p_0\cos[\omega_0(t - s_x x - s_z z)]$. The propagation direction of the wave makes an angle α with the z-axis. Express s_x and s_z in terms of α and c. Give expressions for the wavelength λ as well as the wavelengths λ_x and λ_z along the x- and z-axes.

2a. Consider the linearized equations of continuity and motion for a source-free medium, with $\rho = \rho(z)$ and $K = K(z)$. Apply a temporal and 2D spatial Fourier transform to these equations, from the (x, y, z, t)-domain to the (k_x, k_y, z, ω)-domain.

2b. Eliminate $\tilde{V}_x$ and $\tilde{V}_y$ from the results of 2a and reorganize into $\frac{\partial \tilde{\vec{Q}}}{\partial z} = \tilde{\mathbf{A}}\tilde{\vec{Q}}$, where $\tilde{\vec{Q}} = \begin{pmatrix} \tilde{P} \\ \tilde{V}_z \end{pmatrix}$ and $\tilde{\mathbf{A}}$ is a 2×2 matrix. Give an expression for $\tilde{\mathbf{A}}$.

2c. The eigenvalue decomposition of $\tilde{\mathbf{A}}$ is given by $\tilde{\mathbf{A}} = \tilde{\mathbf{L}}\tilde{\mathbf{\Lambda}}\tilde{\mathbf{L}}^{-1}$, with $\tilde{\mathbf{L}} = \begin{pmatrix} 1 & 1 \\ \frac{k_z}{\omega\rho} & -\frac{k_z}{\omega\rho} \end{pmatrix}$. Give expressions for $\tilde{\mathbf{\Lambda}}$, $\tilde{\mathbf{L}}^{-1}$ and k_z.

2d. Substitute $\tilde{\vec{Q}} = \tilde{\mathbf{L}}\tilde{\vec{D}}$ into $\frac{\partial \tilde{\vec{Q}}}{\partial z} = \tilde{\mathbf{A}}\tilde{\vec{Q}}$. The result can be reworked into $\frac{\partial \tilde{\vec{D}}}{\partial z} = \tilde{\mathbf{B}}\tilde{\vec{D}}$. Give an expression for $\tilde{\mathbf{B}}$.

2e. Define $\tilde{\vec{D}} = \begin{pmatrix} \tilde{P}^+ \\ \tilde{P}^- \end{pmatrix}$. Rewrite $\frac{\partial \tilde{\vec{D}}}{\partial z} = \tilde{\mathbf{B}}\tilde{\vec{D}}$ as a system of scalar equations for $\tilde{P}^+$ and $\tilde{P}^-$. What do these equations mean?

2f. Consider an interface at $z = z_1$ between two homogeneous half-spaces, with medium parameters c_u and ρ_u in the upper half-space, and c_l and ρ_l in the lower half-space. Give the boundary conditions at $z = z_1$ for $\tilde{\vec{Q}}$ and for $\tilde{\vec{D}}$.

2g. Assume a plane wave $\tilde{P}_u^+(z)$ is incident to the interface from above. For the reflected and transmitted waves we write $\tilde{P}_u^-(z_1) = \tilde{R}^+(z_1)\tilde{P}_u^+(z_1)$ and $\tilde{P}_l^+(z_1) = \tilde{T}^+(z_1)\tilde{P}_u^+(z_1)$, respectively. Use the boundary condition for $\tilde{\vec{D}}$ to derive expressions for the reflection and transmission coefficients $\tilde{R}^+(z_1)$ and $\tilde{T}^+(z_1)$.

3a. Substitute $\vec{Q} = P^A(\frac{1}{\rho^B}\nabla P^B) - P^B(\frac{1}{\rho^A}\nabla P^A)$ into the theorem of Gauss $\oint_{\mathbb{S}} \vec{Q} \cdot \vec{n} \mathrm{d}\mathbb{S} = \int_{\mathbb{V}} \nabla \cdot \vec{Q} \mathrm{d}\mathbb{V}$ and use this to derive a general reciprocity theorem between two acoustic states A and B in the space-frequency domain. (Hint: use the Fourier transform of the wave equation derived in 1c for states A and B).

3b. Let $K^A = K^B$ and $\rho^A = \rho^B$. Show how the general reciprocity theorem of 3a can be used to derive reciprocity between a source and a receiver, i.e., $P^B(\vec{r}_A, \omega) = P^A(\vec{r}_B, \omega)$.

3c. Consider again the reciprocity theorem of 3a. Choose for state A the causal Green's state [hence, $P^A(\vec{r}, \omega) \rightarrow G(\vec{r}, \vec{r}_A, \omega)$ etc], and choose for state B the actual state [hence, $P^B(\vec{r}, \omega) \rightarrow P(\vec{r}, \omega)$ etc]. Derive an expression for $P(\vec{r}_A, \omega)$ of the form $P(\vec{r}_A, \omega) = \oint_{\mathbb{S}} (\cdot\cdot) \cdot \vec{n} \mathrm{d}\mathbb{S}$. Discuss the result.

3d. Let $\mathbb{S}$ consist of a horizontal plane $\mathbb{S}_0$ at $z = z_0$ and a half-sphere $\mathbb{S}_1$ with infinite radius in the lower half-space $z > z_0$. Make a picture of the configuration and modify the result of 3c for this configuration.

3e. We use the result of 3d as a starting point for deriving a one-way Rayleigh integral. Substitute $P(\vec{r}, \omega) = P^+(\vec{r}, \omega) + P^-(\vec{r}, \omega)$ and $G(\vec{r}, \vec{r}_A, \omega) = G^-(\vec{r}, \vec{r}_A, \omega)$ at $z = z_0$. You may use the shorter notation $P = P^+ + P^-$ and $G = G^-$.

3f. Assume that the medium is laterally invariant at $z = z_0$. Rewrite the result of 3e as integrals along k_x and k_y, using Parseval's theorem $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(x,y)B(x,y)dxdy = (\frac{1}{2\pi})^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{A}(-k_x,-k_y)\tilde{B}(k_x,k_y)dk_x dk_y$.

3g. Substitute the one-way wave equations $\frac{\partial \tilde{P}^{\pm}}{\partial z} = \mp jk_z \tilde{P}^{\pm}$ and $\frac{\partial \tilde{G}'^-}{\partial z} = jk_z \tilde{G}'^-$ at $z = z_0$. Simplify the result as much as possible.

3h. Substitute again the one-way wave equation $jk_z \tilde{G}'^- = \frac{\partial \tilde{G}'^-}{\partial z}$ at $z = z_0$. Use Parseval's theorem to rewrite the result as integrals along x and y. Discuss the result.