# 'Advanced reflection seismology and seismic imaging' (AES 1560)

## November 7, 2011

1a. Consider the wave equation in spherical coordinates for a homogeneous medium

$$\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} - \frac{\rho}{K}\frac{\partial^2 p}{\partial t^2} = 0, \quad \text{for} \quad r \neq 0.$$

A solution is $p(r,t) = \frac{u(t-r/c)}{r}$. Explain this solution in physical terms. Express $c$ in terms of $\rho$ and $K$.

1b. Show that $p(r,t) = \frac{u(t-r/c)}{r}$ is indeed a solution of the wave equation (for $r \neq 0$).

1c. In Cartesian coordinates the Green's function $g(\vec{r}, \vec{r}_A, t)$ in a homogeneous medium obeys the wave equation

$$\nabla^2 g(\vec{r}, \vec{r}_A, t) - \frac{\rho}{K}\frac{\partial^2 g(\vec{r}, \vec{r}_A, t)}{\partial t^2} = -\rho\delta(\vec{r} - \vec{r}_A)\delta(t).$$

Give the causal solution $g(\vec{r}, \vec{r}_A, t)$ and the anticausal solution $\hat{g}(\vec{r}, \vec{r}_A, t)$ for a homogeneous medium. Also give the relation between these solutions.

1d. Derive the temporal Fourier transforms of $g(\vec{r}, \vec{r}_A, t)$ and $\hat{g}(\vec{r}, \vec{r}_A, t)$. Give the relation between $G(\vec{r}, \vec{r}_A, \omega)$ and $\hat{G}(\vec{r}, \vec{r}_A, \omega)$.

1e. The Kirchhoff-Helmholtz integral is given by

$$P(\vec{r}_A, \omega) = \oint_{\mathbb{S}} \frac{1}{\rho}\left[G(\vec{r}, \vec{r}_A, \omega)\frac{\partial P(\vec{r}, \omega)}{\partial n} - \frac{\partial G(\vec{r}, \vec{r}_A, \omega)}{\partial n}P(\vec{r}, \omega)\right] d\mathbb{S}.$$

Discuss this expression.

1f. Subdivide $\mathbb{S}$ into a horizontal surface $\mathbb{S}_0$ at $z = z_0$ and a hemisphere $\mathbb{S}_1$ in the lower half-space. Derive the Rayleigh-I and Rayleigh-II integrals. Hint: use boundary condition $\partial G/\partial z = 0$ or $G = 0$ at $z = z_0$.

1g. Show that for a homogeneous medium the Rayleigh-II integral can be written as a spatial convolution, according to

$$P(x_A, y_A, z_A, \omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(x_A - x, y_A - y, z_A, z_0, \omega) P(x, y, z_0, \omega) \mathrm{d}x\mathrm{d}y.$$

Give the relation between $W$ and the Green's function $G$.

2a. Consider the linearized equation of continuity

$$\nabla \cdot \vec{v}(x,y,z,t) + \frac{1}{K(z)} \frac{\partial p(x,y,z,t)}{\partial t} = \frac{\partial i_v(x,y,z,t)}{\partial t},$$

and the linearized equation of motion

$$\nabla p(x,y,z,t) + \rho(z) \frac{\partial \vec{v}(x,y,z,t)}{\partial t} = \vec{f}(x,y,z,t).$$

Transform these equations to the space-frequency domain (that is, to the $(x,y,z,\omega)$-domain).

2b. Transform the equations to the wavenumber-frequency domain (that is, to the $(k_x,k_y,z,\omega)$-domain).

2c. Derive the two-way wave equation

$$\frac{\partial \tilde{\vec{Q}}}{\partial z} = \tilde{\mathbf{A}}\tilde{\vec{Q}} + \tilde{\vec{S}}, \quad \text{where} \quad \tilde{\vec{Q}}(k_x,k_y,z,\omega) = \begin{pmatrix} \tilde{P}(k_x,k_y,z,\omega) \\ \tilde{V}_z(k_x,k_y,z,\omega) \end{pmatrix}.$$

2d. The eigenvalue decomposition of $\tilde{\mathbf{A}}$ is $\tilde{\mathbf{L}}\tilde{\mathbf{\Lambda}}\tilde{\mathbf{L}}^{-1}$. Give expressions for $\tilde{\mathbf{L}}$, $\tilde{\mathbf{\Lambda}}$ and $\tilde{\mathbf{L}}^{-1}$.

2e. Substitute $\tilde{\vec{Q}} = \tilde{\mathbf{L}}\tilde{\vec{D}}$ and the eigenvalue decomposition of $\tilde{\mathbf{A}}$ into the two-way wave equation and derive the one-way wave equation

$$\frac{\partial \tilde{\vec{D}}}{\partial z} = \tilde{\mathbf{B}}\tilde{\vec{D}} + \tilde{\vec{S}}'(z), \quad \text{where} \quad \tilde{\vec{D}}(k_x,k_y,z,\omega) = \begin{pmatrix} \tilde{P}^+(k_x,k_y,z,\omega) \\ \tilde{P}^-(k_x,k_y,z,\omega) \end{pmatrix}.$$

2f. Rewrite this one-way wave equation into two coupled scalar one-way wave equations for $\tilde{P}^+$ and $\tilde{P}^-$. Show that for a homogeneous, source-free medium, these equations decouple into independent one-way wave equations for $\tilde{P}^+$ and $\tilde{P}^-$.

2g. From the one-way wave equation for $\tilde{P}^+$, derive the following expression for forward wave field extrapolation:

$$\tilde{P}^+(k_x,k_y,z_A,\omega) = \tilde{W}^+(k_x,k_y,z_A,z_0,\omega)\tilde{P}^+(k_x,k_y,z_0,\omega).$$

2h. Discuss the relation between the expressions in questions 1g and 2g.

3a. Suppose there is a point source at $(x, y, z) = (0, 0, z_s)$ in a homogeneous subsurface and that we measure the response of this at the free surface $z = z_0$. Would you measure $P(x, y, z_0, \omega)$ or $V_z(x, y, z_0, \omega)$ or both? Why? What type of instruments do you need for these measurements?

3b. Give an expression by which you can obtain the upgoing wave field $\tilde{P}^-(k_x, k_y, z_0, \omega)$ from these measurements.

3c. Next we want to extrapolate the wave field $P^-(x, y, z_0, \omega)$ from the acquisition level ($z_0$) to the source level ($z_s$). Explain why we cannot use the expression derived in question 1g.

3d. The operator we need for inverse extrapolation we call $F^-(x, y, z_s, z_0, \omega)$. Give a formal relation between $F^-(x, y, z_s, z_0, \omega)$ and the forward operator $W^-(x, y, z_0, z_s, \omega)$.

3e. Transform the answer from 3d from the space-frequency domain to the wavenumber-frequency domain. Give expressions for $\tilde{W}^-(k_x, k_y, z_0, z_s, \omega)$ and $\tilde{F}^-(k_x, k_y, z_s, z_0, \omega)$.

3f. Give an expression for a stable approximation $\langle \tilde{F}^-(k_x, k_y, z_s, z_0, \omega) \rangle$. Derive from this an expression for $\langle F^-(x, y, z_s, z_0, \omega) \rangle$.

3g. Inverse extrapolation is now defined as $\langle F^-(x, y, z_s, z_0, \omega) \rangle * P^-(x, y, z_0, \omega)$, where $*$ stands for spatial convolution. Recall from 3a to 3c that $P^-(x, y, z_0, \omega)$ is the upgoing field at $z_0$ of a point source at $(0, 0, z_s)$. Explain how the inverse extrapolation result is related to this point source.