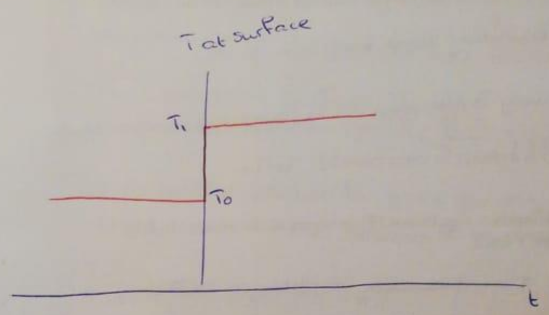
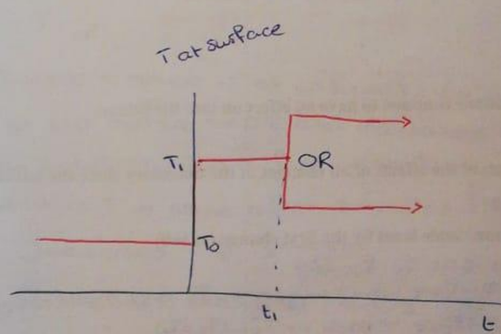
3. Superposition - *time*-varying boundary conditions

We have a variety of 1D unsteady conduction solutions with BC imposed at time 0 and maintained indefinitely thereafter.



Superposition allows us to solve for *time-varying* boundary conditions



BUT it only works for dimensionless temperature $[(T-T_0)/(T_1-T_0)]$, *not* $[(T_1-T)/(T_1-T$

Approach:

a. Consider the nature of the differential equation (pde)

If some function $\theta_1(x,y,z,t)$ satisfies the pde
and some other function $\theta_2(x,y,z,t)$ also satisfies the pde,
THEN any linear combination of $\theta_1 + \theta_2$ also satisfies the pde

The trick in superposition is to find a linear combination of functions that satisfies the pde and also satisfies the time-varying *boundary conditions*.

b. Consider the boundary conditions as *change* in boundary constraints.
All unsteady conduction problems are the result in a change in boundary conditions.

for $t < 0$, $T = T_0$ at boundary (and everywhere else!)

for $t \geq 0$, T at boundary is changed to T_1, and maintained there

the change at the boundary is a change in temperature by $T_1 - T_0$

c. Consider later changed boundary conditions (T) in terms of the *change* in BC, not the value of the new T itself

suppose at time $= t_1$, T at boundary is changed to T_2

the *change* is $T_2 - T_1$

d. All changes in boundary conditions continue to have an effect on into the future, even after later changes

The complete solution is the sum of the effects of all changes at the boundary since the initial, uniform state

But the dimensionless temperature scale is set by the <u>first</u> change (at t=0)

Example: Suppose solid is at T_0 for $t \leq 0$

at $t = 0$, T at surface is changed to T_1 and maintained there

at $t = t_1$, T at surface → back to T_0 ($T_2 = T_0$)

<u>change</u> at $t = 0$ is $(T_1 - T_0)$ at boundary

dimensionless temperature $= \frac{T - T_0}{T_1 - T_0}$ for rest of solution

<u>change</u> at $t = t_1$ is $(T_0 - T_1)$ at boundary

dimensionless temperature change is $\frac{T_0 - T_1}{T_1 - T_0}$ $= -1$

<u>let</u> $\theta(x,y,z,t)$ be solution for $\left(\frac{T - T_0}{T_1 - T_0}\right)$ for a solid initially at T_0, with a surface at T_1

THEN, for any time after t_1

$$\left(\frac{T - T_0}{T_1 - T_0}\right) = [\underbrace{\theta(x,y,z,t)}_{\text{effect of 1st change}} - \underbrace{\theta(x,y,z,(t - t_1))}_{\text{effect of 2nd change}}]$$

More examples suppose solid is at T_0 at time 0

at $t=0$, T at surface $\rightarrow T_1$

at $t=t_1$, T at surface $\rightarrow T_2$

at $t=0$, change in boundary condition is (T_1-T_0)

dimensionless temperature: $\left(\frac{T-T_0}{T_1-T_0}\right)$ ← for whole problem solution

at $t=t_1$, change at surface is (T_2-T_1)

dimensionless change is $\left(\frac{T_2-T_1}{T_1-T_0}\right)=C$

let Θ be the solution for given geometry with only the change at $t=0$ (i.e., solution on chart)

$$\left(\frac{T-T_0}{T_1-T_0}\right)=\Theta(x,y,z,t) \text{ up to } t_1$$

$$=\Big[\underbrace{\Theta(x,y,z,t)}_{\text{effect of change at } t=0}+\underbrace{C\,\Theta(x,y,z,(t-t_1))}_{\text{effect of 2nd change}}\Big] \text{ for } t>t_1$$

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suppose a sphere is at uniform $T=100°C = T_0$ at $t=0$

at $t=0$, surface $\rightarrow 0°C$ and maintained there

at $t=t_1=2\,min=120s$, surface $\rightarrow 200°C$

What is T at centre after 3 minutes from start?

Dimensionless Temp is $\frac{T-100}{0-100}$

Dimensionless change at 2 min is $\frac{200-0}{0-100}=-2$

at $t=3$ min

$$\left(\frac{T-T_0}{T_1-T_0}\right)=\left(\frac{T-100}{0-100}\right)=\Big[\Theta(r/R=0,180s)-2\,\Theta(r/R=0,60s)\Big]$$

look up in BSLK Fig 11S-3

A worked problem on superposition.

a. A long (infinitely long) cylinder of aluminum (properties below), 0.2 m in diameter, is i
At time t=0, its surface is suddenly raised to 100°C. What are the the temperature of th
center after 22 s?

b. What is the temperature distribution in the cylinder at this time?

Properties of aluminum

$\rho = 2700$ kg/m^3 $k = 230$ W/(m K) $C_p = 938$ J/(kg K)

a) $\alpha = \frac{k}{\rho c_p} = \frac{230}{2700 \cdot 938} = 9{,}08 \cdot 10^{-5}$ m^2/s

at $t = 22s$, $\frac{\alpha t}{R^2} = \frac{9{,}08 \cdot 10^{-5} \cdot 22}{(0{,}1)^2} = 0{,}2$

$\frac{T - T_0}{T_1 - T_0} = 0{,}5$; $\frac{T - 0}{100 - 0} = 0{,}5 \rightarrow T = 50°C$

b) T distribution is given by BSLK Fig 11.5-3 for $\frac{\alpha t}{R^2} = 0{,}2$

c. At t = 22 s, the surface temperature of the cylinder is reduced again to 0°C. What is the temperature of the cylinder at its center after a total of 44 s since the start, i.e. 22 s after the second change in temperature?

d. What is the temperature distribution inside the cylinder after a total of 44s since the start of the experiment?

Note that this problem is easier because the time lined up exactly with the curves on the BSL charts; in cases, you should be prepared to do interpolation. You should be able to solve for any initial (up or do change and sequence in subsequent changes in T, and for other shapes.

c) change at $t=0$ (100-0)

dimensionless temp. is $\left(\frac{T-0}{100-0}\right)$

dimensionless change at t_1 is

$$\left(\frac{0-100}{100-0}\right) = -1$$

For $t = 44s > 22s$

$$\frac{T-T_0}{100-0} = \Theta(r=0, 44s) - 1\,\Theta(r=0, 22s)$$

where Θ ~~is~~ comes for BSLk Fig 11.5-2

(at 44s, $\frac{\alpha t}{R^2} = 0{,}4$)

$$\frac{T-T_0}{T_1-T_0} = 0{,}84 - 0{,}5 = 0{,}34 = \frac{T-0}{100-0}$$

$$T = 34°$$

d) anywhere in the ~~g~~ cylinder,

$\frac{T-T_0}{T_1-T_0}$ is the difference between curves

for $\frac{\alpha t}{R^2} = 0{,}4$ and $\frac{\alpha t}{R^2} = 0{,}2$

A spreadsheet to illustrate superposition

The spreadsheet "**superposition semi-infinite slab.xls**" is available on the Blackboard site. It solves for temperature distribution into a semi-infinite solid. It allows for up to five changes in boundary temperature, at times t1, t2, t3, t4 and t5, after the initial change at t=0. The initial temperature change can be either up or down in T, but the dimensionless temperature change at t=0 is by definition 1.

he user specifies that temperature at the boundary takes the values in column C at times in column B. Note that it is the dimensionless change in temperature, in column D, that enters into the calculations. The graph plots temperature as a function of position in the solid at the time specified in cell E2. Length here is in m, and time in units ((4 α t)=[4 (k/(ρ C_p)] t). Since α is often of order 10^{-6}, one unit of time is quite a long time.

e spreadsheet's calculations and layout illustrate the superposition principle. Column F lists positions from 0 to 2. For the time given in cell E2, for each position, the influence of each temperature change since t=0 is given in columns H, I, J, K and L, respectively, for the changes at times t1 to t5. Note that if he current time (E2) is before a given change, that change has no effect, because it hasn't happened yet. Thus if the time in cell E2 is less than all changes but that at t=0, all of columns H to L are blank; only ie initial condition (column G) has an effect on temperature. For the given time, the temperature at any ven position (column M) is the sum of the effects of all the changes in boundary condition since the art. The dimensional T (column N) is computed from the dimensionless T and the initial T change. use the solution for unsteady conduction in a semi-infinite medium is based on a well-known function e error function), the spreadsheet can calculate the effect of each change in boundary conditions olumns G to L) analytically. For the more complicated cases of conduction in a finite-width slab, etc., would have to look up the dimensionless temperature at each position and time from the chart.

ole (initial state of the spreadsheet on the Blackboard site)

se at time t=0 the solid is at a uniform temperature 0°C. At time t=0 the surface is raised to 50°C. sets the dimensionless temperature scale, which is [(T-0)/(50-0)]. The change at t=0 is 1 ensionless unit. Then at t = 0.5 surface temperature is returned to the initial temperature. The change 50°), or (-1) in dimensionless units. Then at t=1 it surface temperatures is increased to 100°; the ge in boundary condition is +2 dimensionless units. At t = 1.5 the surface is returned to 0°; a change surface temperature of (-2) dimensionless units. Then at time t=2 it is increased to 150° (3 isionless units), and returned to 0° at t=2.5 (a change of (-3) dimensionless units).

hanges in dimensionless temperature at the boundary are then (See column D)

=0	-1 at t=0.5 (i.e., T back to its initial value)
=1	-2 at t = 1.5 (i.e., T back to its initial value)
=2	-3 at t=2.5 (i.e., T back to its initial value)

7, all six changes in the boundary condition have an effect on each position's temperature. They ed in columns G to L; column M, with the sum of all those changes, is the local temperature, and N calculates dimensional temperature.

illustrated in the spreadsheet:

change in temperature defines the dimensionless temperature for all subsequent changes. The first n temperature therefore is by definition 1. If the initial change in surface temperature is a in T, the initial dimensionless change is still 1, but dimensionless temperature is defined so that ive if temperature is less than the initial temperature.

ture at any given time is the sum of the effect of all the changes in boundary condition since the .

conditions continues to have an effect indefinitely thereafter.

t has not been made yet has no effect (of course).

Not changes that haven't happened yet! (of course)