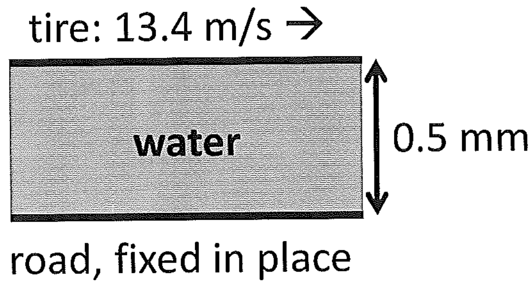
tn4780ta 2013-14
Part 1 Final Examination - 28 Jan. 2014

Write your solutions *on your answer sheet*, not here. In all cases ***show your work.***
To avoid any possible confusion,
state the equation numbers and figure numbers of equations and figures you use.
Beware of unnecessary information in the problem statement.

1. When an automobile tries to stop in wet weather, a film of water between the automobile tires and the road can greatly reduce its ability to decelerate. Suppose an automobile is traveling at 48 km/hr (13.4 m/s). Its tires are "bald," and the surface of the street is absolutely flat, so in effect the water film (properties given below) is a flat film of thickness 0.5 mm, between a moving, flat surface (the tires) and a fixed flat surface (the road). Assume the four tires, each, have a surface area on the road of 15 cm x 30 cm (i.e., total area for all four = 0.18 m^2). Ignore any edge effects around the edge of the tires where the tires and the road meet.
   a. Assuming laminar flow applies, what force do the tires as a group exert to slow down the car at the first instant, when the car is going 13.4 m/s? If the car weighs 900 kg, what is its initial rate of deceleration?
   b. Is the assumption of laminar flow in part (a) justified? If you are unable to answer this question quantitatively, tell how you would obtain an answer.

   (In fact, roads are roughened and tires are not flat, in order to prevent this sort of film from forming. That is why it's so important to replace bald tires.)
   (20 points)



properties of water
$\mu = 0.001$ Pa s $\qquad \rho = 1000$ kg/m^3

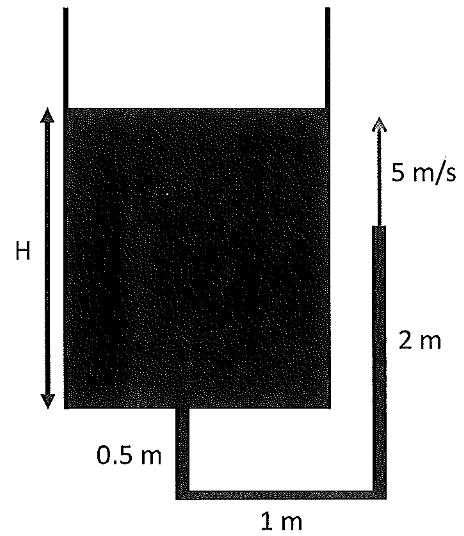
2. An insect is blown off from the top of a tall building into the air. It doesn't have wings and can't fly, but as it falls, it spreads out is legs, giving it drag equivalent to a sphere 5 mm in diameter. Averaged over this diameter, its density is 125 kg/m^3. Treating the insect as a sphere of density 125 kg/m^3, what is its steady velocity as it falls through the air? The properties of air are given below.

   Don't worry about whether the simplifications I make are correct; just solve the problem as posed.
   (25 points)

properties of air
$\mu = 1.75 \times 10^{-5}$ Pa s $\qquad \rho = 1.26$ kg/m^3

3. Water flows out from a tall tank, through an abrupt, sharp (not rounded) entrance into a pipe 2 cm in diameter; the pipe goes down 0.5 m, then right 1 m, then up 2 m; there are two sharp 90° elbows along the way (see picture below). Water shoots out the open end of the pipe with velocity 5 m/s. The roughness of the pipe wall is 0.005 cm. The properties of water are given in problem 1.
   a. What is the height of the water in the tank, H?
   b. The engineer working on this problem is not very sure about the roughness in the pipe. He thinks the pipe might be *much* rougher, increasing the drag in the pipe by a factor of 2 or more. How much difference would that make to the value of H? Briefly justify your answer. If you're not able to answer part (a), describe how you *would* answer this question.

   (35 points)



4. Suppose a tube of radius R is filled with Newtonian fluid in steady flow. The viscosity of this fluid is not uniform in the tube, however, but varies with radial position r as

$$\mu = \frac{1}{\left(A+B\left(\frac{r}{R}\right)\right)}$$

   where A and B are constants (This equation is related to a model for flow of polymer solutions through narrow pores, where the polymer avoids the pore wall.) This problem is similar to the problem of a Newtonian fluid in a tube in solved in BSL Section 2.3 (pages attached to end of this exam).
   a. What is the last equation in that derivation that can be used directly in this problem?
   b. Starting with that equation, derive the velocity profile in the tube, $v_z(r)$.

   (20 points)

We consider then the steady laminar flow of a fluid of constant density ρ in a "very long" tube of length L and radius R; we specify that the tube be "very long" because we want to assume that there are no "end effects"; that is, we ignore the fact that at the tube entrance and exit the flow will not necessarily be parallel everywhere to the tube surface.

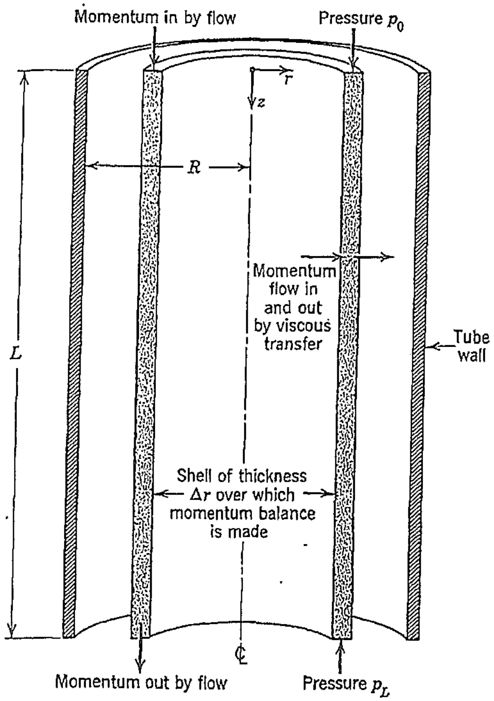


Fig. 2.3-1. Cylindrical shell of fluid over which momentum balance is made to get the velocity profile and the Hagen-Poiseuille formula for the volume rate of flow.

We select as our system a cylindrical shell of thickness Δr and length L (see Fig. 2.3-1), and we begin by listing the various contributions to the momentum balance in the z-direction:

rate of momentum in across cylindrical surface at r	$(2\pi r L\tau_{rz})\|_r$	(2.3-1)

rate of momentum out across cylindrical surface at $r + \Delta r$	$(2\pi r L\tau_{rz})\|_{r+\Delta r}$	(2.3-2)
rate of momentum in across annular surface at $z = 0$	$(2\pi r\Delta r\, v_z)(\rho v_z)\|_{z=0}$	(2.3-3)
rate of momentum out across annular surface at $z = L$	$(2\pi r\Delta r\, v_z)(\rho v_z)\|_{z=L}$	(2.3-4)
gravity force acting on cylindrical shell	$(2\pi r\Delta r L)\rho g$	(2.3-5)
pressure force acting on annular surface at $z = 0$	$(2\pi r\Delta r)p_0$	(2.3-6)
pressure force acting on annular surface at $z = L$	$-(2\pi r\Delta r)p_L$	(2.3-7)

Note once again that we take "in" and "out" to be in the positive direction of the axes.

We now add up the contributions to the momentum balance:

$$(2\pi r L\tau_{rz})|_r - (2\pi r L\tau_{rz})|_{r+\Delta r} + (2\pi r\,\Delta r\rho v_z^2)|_{z=0} - (2\pi r\,\Delta r\rho v_z^2)|_{z=L} + 2\pi r\,\Delta r L\rho g + 2\pi r\,\Delta r(p_0 - p_L) = 0 \quad (2.3\text{-}8)$$

Because the fluid is assumed to be incompressible, v_z is the same at $z = 0$ and $z = L$, hence the third and fourth terms cancel one another. We now divide Eq. 2.3-8 by $2\pi L\,\Delta r$ and take the limit as Δr goes to zero; this gives

$$\lim_{\Delta r \to 0}\left(\frac{(r\tau_{rz})|_{r+\Delta r} - (r\tau_{rz})|_r}{\Delta r}\right) = \left(\frac{p_0 - p_L}{L} + \rho g\right)r \quad (2.3\text{-}9)$$

The expression on the left side is the definition of the first derivative. Hence Eq. 2.3-9 may be written as

$$\frac{d}{dr}(r\tau_{rz}) = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{L}\right)r \quad (2.3\text{-}10)$$

in which[1] $\mathscr{P} = p - \rho g z$. Eq. 2.3–10 may be integrated to give:

$$\tau_{rz} = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{2L}\right) r + \frac{C_1}{r} \tag{2.3–11}$$

The constant C_1 must be zero if the momentum flux is not to be infinite at $r = 0$. Hence the momentum flux distribution is

$$\boxed{\tau_{rz} = \left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{2L}\right) r} \tag{2.3–12}$$

This distribution is shown in Fig. 2.3–2.

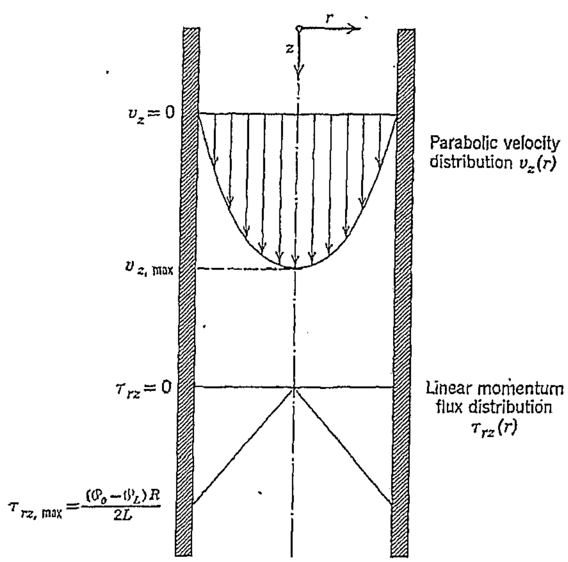


Fig. 2.3–2. Momentum flux and velocity distributions in flow in cylindrical tubes.

Newton's law of viscosity for this situation is

$$\tau_{rz} = -\mu \frac{dv_z}{dr} \tag{2.3–13}$$

[1] The quantity $\mathscr{P}$ represents the combined effect of static pressure and gravitational force. To allow for other flow orientations, $\mathscr{P}$ may be defined more generally as $\mathscr{P} = p + \rho g h$, where h is the distance *upward* (that is, in the direction opposed to gravity) from any chosen reference plane.

Substitution of this relation into Eq. 2.3–12 then gives the following differential equation for the velocity:

$$\frac{dv_z}{dr} = -\left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{2\mu L}\right) r \tag{2.3–14}$$

Integration of this gives

$$v_z = -\left(\frac{\mathscr{P}_0 - \mathscr{P}_L}{4\mu L}\right) r^2 + C_2 \tag{2.3–15}$$

Because of the boundary condition that v_z be zero at $r = R$, the constant C_2 has the value $(\mathscr{P}_0 - \mathscr{P}_L)R^2/4\mu L$. Hence the velocity distribution is

$$\boxed{v_z = \frac{(\mathscr{P}_0 - \mathscr{P}_L)R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right]} \tag{2.3–16}$$

This result tells us that the velocity distribution for laminar, incompressible flow in a tube is parabolic. (See Fig. 2.3–2.)

Once the velocity profile has been established, various derived quantities are easily calculated:

(i) The *maximum velocity* $v_{z,\max}$ occurs at $r = 0$ and has the value

$$v_{z,\max} = \frac{(\mathscr{P}_0 - \mathscr{P}_L)R^2}{4\mu L} \tag{2.3–17}$$

(ii) The *average velocity* $\langle v_z \rangle$ is calculated by summing up all the velocities over a cross section and then dividing by the cross-sectional area:

$$\langle v_z \rangle = \frac{\int_0^{2\pi}\int_0^R v_z r\, dr\, d\theta}{\int_0^{2\pi}\int_0^R r\, dr\, d\theta} = \frac{(\mathscr{P}_0 - \mathscr{P}_L)R^2}{8\mu L} \tag{2.3–18}$$

The details of the integration are left to the reader. Note that $\langle v_z \rangle = \frac{1}{2} v_{z,\max}$.

(iii) The *volume rate of flow* Q is the product of area and average velocity; thus

$$Q = \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R^4}{8\mu L} \tag{2.3–19}$$

This rather famous result is called the Hagen-Poiseuille[2] law in honor of the two scientists[3,4] credited with its formulation. It gives the relationship

[2] Pronounce Poiseuille as "Pwah-zə'-yuh," in which ə is roughly the same as the "oo" in the American pronunciation of "book".
[3] G. Hagen, *Ann. Phys. Chem.*, **46**, 423–442 (1839).
[4] J. L. Poiseuille, *Compte Rendus*, **11**, 961 and 1041 (1840); **12**, 112 (1841).