III. Shell Momentum Balances

A. recap of concepts so far

B. general form of momentum balance BSLK 42

(rate of momentum in by convection) − (rate of momentum out by convection)

+ (rate of momentum in by shear stress) − (rate of momentum out by shear stress)

+ (body forces (gravity) acting on fluid) = (accumulation of momentum (acceleration))

↳ = 0 at steady state

C. outline of shell momentum balance approach
(BSL ch. 2) (4 BSLk P.43)

Procedure: (*cf.* BSL section 2.1)

1. SELECT COORDINATE SYSTEM; DEFINE CONTROL VOLUME
2. STATE BOUNDARY CONDITIONS *
3. PERFORM MOMENTUM BALANCE **
4. THICKNESS → 0 (→ dif. eq. for τ)

a) (optional): solve dif. eq. for τ, apply b.c. - IF b.c. applies to τ alone

5. RELATE τ TO dv/dx (apply constitutive equation)
6. SOLVE DIF. EQ. FOR v; APPLY B.C. *
   a) (optional) COMPUTE w (mass rate of flow) or Q (volume rate of flow), etc.

* BOUNDARY CONDITIONS (*cf.* BSL section 2.1) also BSLk 48.

1. SPECIFY v AT SOLID SURFACE 'no-slip' BC.
   1a) FLUID v = SOLID VELOCITY AT SOLID WALL
2. SPECIFY τ AT FLUID SURFACE
   2a) IN LIQUID, $\tau = 0$ AT GAS I.F.
   2b) τ, v CONTINUOUS ACROSS LIQUID/LIQUID I.F.
3. τ, v NOT INFINITE ANYWHERE IN REGION OF INTEREST

"ALL BOUNDARY CONDITIONS ARISE FROM NATURE"
(i.e., from problem statement)

** ELEMENTS OF MOMENTUM BALANCE

MOMENTUM FLUX (α area); called "ϕ" tensor in BSL (sect. 1.7) (BSLK Sect 1.3)

1. CONVECTION OF MOMENTUM THRU SURFACE (ρvv)
2. SHEAR STRESS τ ON SURFACE ("molecular transport of momentum")
3. PRESSURE PRESSING INWARD ON SURFACE - p

MOMENTUM "GENERATION" or "SOURCE" (α volume)

4. BODY FORCES WITHIN VOLUME

MOMENTUM ACCUMULATION (α volume) (not at steady state!)

5. ACCELERATION OF SYSTEM MASS - $\partial(\rho\, v)/\partial t$

1. **Four aspects can vary from one problem to another**

- geometry (coordinate system)
- elements in momentum balance
- boundary conditions
- constitutive equation
  (relation between τ and dv/dy)

D. Examples

1. **Falling film**
   a. **Newtonian fluid - BSL Sect. 2.2**
      i. **Newtonian fluid with position-dependent viscosity - BSL Ex. 2.2-2**

Since geometry and momentum balance are unchanged (see C.1 above),
can jump directly to Eq. 2.2-13 (2.2-26)
Don't sweat the algebraic details

   b. **Bingham plastic** - to be covered on homework

2. **flow through circular tube**

   a. **Newtonian fluid - BSL Sect. 2.3**

      i. **an aside: flow of ideal gas in tube** (attached)

equations used in natural gas engineering
not required material for this course

3.5

b. Bingham plastic in tube - BSL Ex. 2.3-2 (*first edition of BSL*)

Notes:

- only the relation between τ_{rz} and (dv_z/dr) changes from Newtonian fluid in tube; geometry, momentum balance, boundary conditions are unchanged.
  Therefore everything thru Eq. 2.3-13* still applies

- Recall definition of Bingham plastic:

$$\tau_{rz} = -\mu_o \frac{dv_z}{dr} + \tau_o \qquad \tau_{rz} \geq \tau_o$$

$$\frac{dv_z}{dr} = 0 \qquad -\tau_o \leq \tau_{rz} \leq \tau_o$$

$$\tau_{rz} = -\mu_o \frac{dv_z}{dr} - \tau_o \qquad \tau_{rz} \leq -\tau_o$$

- Drawing a picture of $\tau_{rz}(r)$ is essential (cf. Fig. 2.3-3*)

i. calculating Q and w for Bingham plastic in tube

- calculate $\tau_R \equiv \dfrac{\Delta P\ R}{2\ L}$ at the wall

- compare τ_R, τ_0:

  - if $\tau_R \leq \tau_o$, Q = 0, and w = 0. Don't use Eq. 2.3-30*!

  - if $\tau_R \geq \tau_o$, Q given by Eq. 2.3-30. w = Q ρ

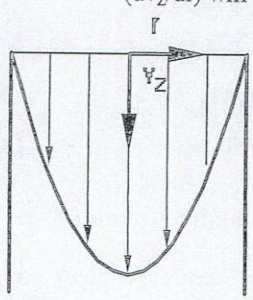
(* - Eq. no. 2.3-12 from 1[st] ed. of BSL)

<u>**tips for homework:**</u>

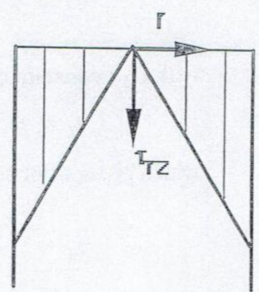
c. **Power-Law Fluid**

Key points in this derivation are as follows:

1) Because the system geometry, elements in the momentum balance, and boundary conditions are the same as for flow of a Newtonian fluid in a tube (BSL Section, 2.3), we can skip directly to Eq. 2.3-13, just before Newton's law is introduced in Section 2.3.
2) It is important to recall the mathematical definition of absolute value:
$$|x| = x \quad \text{if } x \geq 0$$
$$= -x \quad \text{if } x \leq 0. \qquad [1]$$
3) Recall that one cannot take arbitrary, fractional powers and roots of negative numbers (unless one is working with imaginary numbers).
4) Due to points (2) and (3), it is important to identify from the start the sign of (dv_z/dr). Of course, deriving $v_z(r)$ is the point of the exercise, so one has to sketch out the expected shape of $v_z(r)$ to guess in advance what the sign of (dv_z/dr) will be.



Expect
v_z = max at r = 0
v_z = 0 at r = R
therefore $(dv_z/dr) < 0$
KNOW (from BSL 2.3-13)
$\tau_{rz} > 0$



BSL Eq. 2.3-13, for flow of any fluid in a tube:

$$\tau_{rz} = \left(\frac{P_o - P_L}{2L}\right) r\ . \qquad [2]$$

For a power-law fluid,

$$\tau_{rz} = -\ m \left|\frac{dv_z}{dr}\right|^{n-1} \frac{dv_z}{dr} \qquad [3]$$

Eventually we want to get rid of the absolute value. As a first step, we need to combine both derivative terms within the absolute value. Since $(dv_z/dr) < 0$ in this case (see diagram above), $(dv_z/dr) = -\,|\,dvz/dr\,|$; therefore

$$\tau_{rz} = -\ m \left|\frac{dv_z}{dr}\right|^{n-1} (-1) \left|\frac{dv_z}{dr}\right| = m \left|\frac{dv_z}{dr}\right|^{n}. \qquad [4]$$

Combining with Eq. [2],

$$\left|\frac{dv_z}{dr}\right|^{n} = \left(\frac{P_o - P_L}{2Lm}\right) r\ . \qquad [5]$$

Both sides of this equation are positive; therefore we can take the nth root of both sides:

$$\left|\frac{dv_z}{dr}\right| = \left(\frac{P_o - P_L}{2Lm}\right)^{1/n} r^{1/n} . \qquad [5]$$

Now for the final time we use the definition of absolute value. Since $(dv_z/dr) < 0$ in this case, $(dv_z/dr) = -\,|\,dvz/dr\,|$, and

$$\frac{dv_z}{dr} = -\left(\frac{P_o - P_L}{2Lm}\right)^{1/n} r^{1/n} . \qquad [6]$$

If one were less careful about handling the absolute values in this derivation, one would end up with the wrong sign upon arriving at Eq. [6]. Integrating Eq. [6] gives

$$v_z = -\left(\frac{P_o - P_L}{2Lm}\right)^{1/n} \frac{1}{[1+(1/n)]} r^{[1+(1/n)]} + C_2 \qquad [7]$$

with C_2 a constant of integration. The final boundary condition is

$$v_z = 0 \quad \text{at } r = R \qquad [8]$$

which gives, after some algebraic manipulation,

$$v_z = \left(\frac{P_o - P_L}{2Lm}\right)^{1/n} \frac{R^{[1+(1/n)]}}{[1+(1/n)]}\left(1-\left(\frac{r}{R}\right)^{[1+(1/n)]}\right). \qquad [9]$$

Note that if $n = 1$ and $m = \mu$, this is the same as BSL Eq. 2.3-18 for a Newtonian fluid.

$$Q = \int_0^R 2\pi r v_z dr \qquad [10]$$

$$Q = 2\pi\left(\frac{P_o - P_L}{2Lm}\right)^{1/n} \frac{R^{[1+(1/n)]}}{[1+(1/n)]}\int_0^R r\left(1-\left(\frac{r}{R}\right)^{[1+(1/n)]}\right)dr \qquad [11]$$

$$Q = 2\pi\left(\frac{P_o - P_L}{2Lm}\right)^{1/n} \frac{R^{[1+(1/n)]}}{[1+(1/n)]}\left(\frac{R^2[1+(1/n)]}{2[3+(1/n)]}\right) \qquad [12]$$

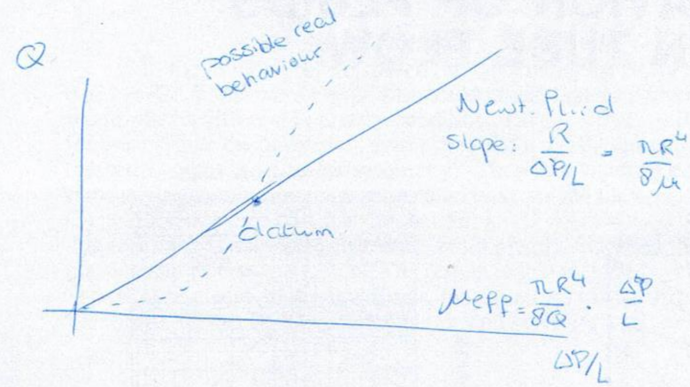
$$Q = \pi\left(\frac{P_o - P_L}{2Lm}\right)^{1/n} \frac{R^{[3+(1/n)]}}{[3+(1/n)]} = \frac{\pi}{(2m)^{1/n}} \frac{R^{[3+(1/n)]}}{[3+(1/n)]}\left(\frac{P_o - P_L}{L}\right)^{1/n} . \qquad [13]$$

Note that if $n = 1$ and $m = \mu$, this is the same as BSL Eq. 2.3-21 for a Newtonian fluid.

Note that while $Q \sim \Delta P/L$ for a Newtonian fluid, $Q \sim (\Delta P/L)^{1/n}$ for a power-law fluid.

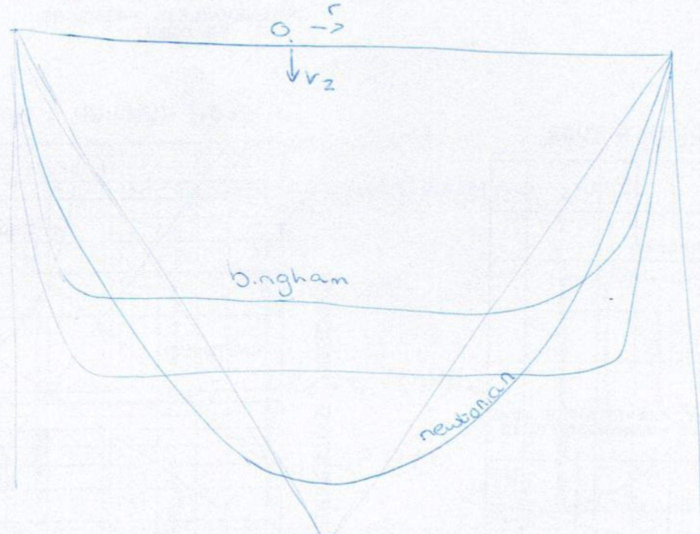
d. Definition of effective viscosity for tube flow

The "effective viscosity" of a non-Newtonian fluid in tube flow is the viscosity of a hypothetical Newtonian fluid that has the same Q as the given fluid in a tube with the same R and $\frac{\Delta P}{L}$.



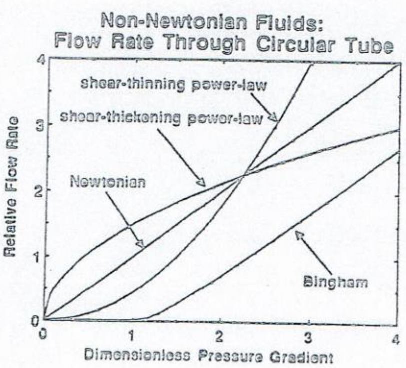
e. Summary of fluid behavior in tube flow

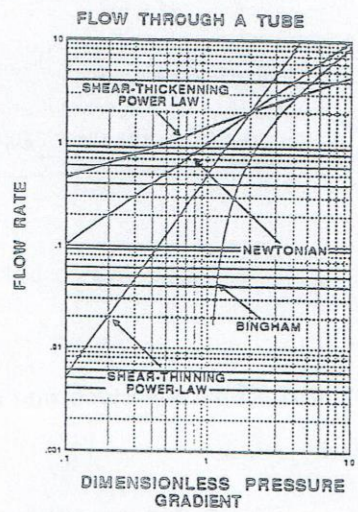
velocity profiles:

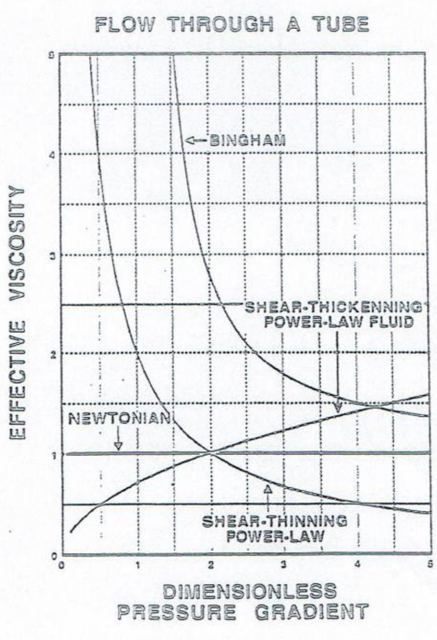


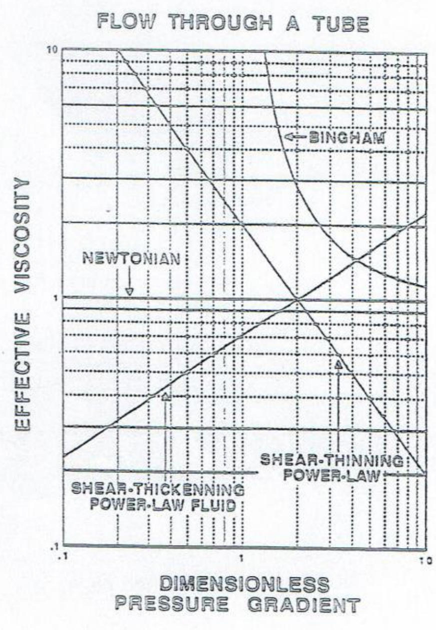
flow rates - on attached page

SUMMARY OF BEHAVIOR OF FLUIDS IN TUBE FLOW









3.10

3. rectangular slit

a. Newtonian fluid - done on homework; BSL p. ~~62~~ 63 BSLK p.71

i. a note on boundary conditions

In the homework, we solved for $v_z(x)$ using the BC (1) $v_z = 0$ at $x = B$ and (2) $v_z = 0$ at $x = -B$. If one were clever, one could notice that the symmetry of the problem about $x = 0$ implies a different boundary condition: (3) $\tau_{xz} = 0$ at $x = 0$. (The justification is as follows: Since the problem is symmetric about $x = 0$, how can there by any momentum transport across the plane of symmetry? Therefore τ_{xz} must be zero at this plane, at $x = 0$.) For the Newtonian flow, then one could have solved for $v_z(x)$ for $x \geq 0$ using BC (1) and (3), and used symmetry to argue that, for $x < 0$, $v_z(x) = v_z(-x)$. For the Newtonian flow this approach is not really necessary, but it greatly simplifies the solution for Bingham plastics and power-law fluids. The reason is that in a slit τ_{xz} changes sign at $x = 0$, and for both Bingham and power-law fluids, the equations for τ_{xz} differ for $\tau_{xz} < 0$ and $\tau_{xz} > 0$. Therefore we use BC (1) and (3) below and solve for $x \geq 0$ only.

b. Bingham plastic

Since the geometry and momentum balance are the same is in the homework solution for Newtonian flow, we can jump directly to "Eq. II" on the homework solution set:

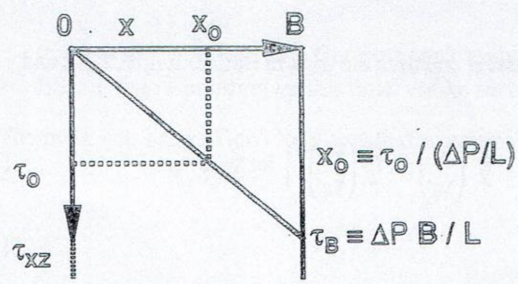
$$\tau_{xz} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L}\, x + C_1 \,. \qquad [1]$$

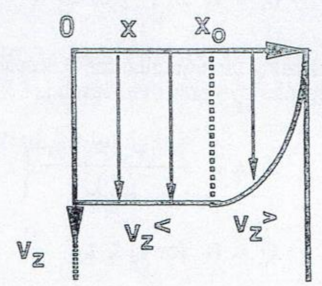
We limit our consideration to $0 \leq x \leq B$ and use BC (3): $\tau_{xz} = 0$ at $x = 0$. This implies

$$C_1 = 0$$

$$\tau_{xz} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L}\, x \,. \qquad [2]$$

At this point it pays to sketch τ_{xz} and the expected v_z profile.





We define x_o is the location where $\tau_{xz} = \tau_o$. That is, 3.11B

$$x_0 \equiv \frac{\tau_0}{\left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L}\right)} \quad . \qquad [3]$$

For $x < x_0$, $t_{xz} < \tau_0$ and therefore $dv_z/dx = 0$, according to the Bingham plastic equation. For $x \geq x_0$,

$$\tau_{xz} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L}\, x = -\mu_0 \frac{dv_z}{dx} + \tau_0 \qquad \text{for } x \geq x_0 \ . \qquad [4]$$

Rearranging and integrating gives

$$v_z^{>} = -\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{\mu_0 L} \frac{x^2}{2} + \frac{\tau_0}{\mu_0} x + C_1 \qquad \text{for } x \geq x_0 \ . \qquad [5]$$

BC (1), $v_z = 0$ at $x = B$, implies

$$C_1 = \frac{\mathcal{P}_0 - \mathcal{P}_L}{\mu_0 L} \frac{B^2}{2} - \frac{\tau_0}{\mu_0} B \qquad [6]$$

$$v_z^{>} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)\, B^2}{2\, \mu_0\, L} \left(1 - \left(\frac{x}{B}\right)^2\right) - \frac{\tau_0\, B}{\mu_0} \left(1 - \frac{x}{B}\right) \qquad \text{for } x \geq x_0 \qquad [7]$$

For $x < x_0$, v_z is given by Eq. [7] with $x = x_0$:

$$v_z^{<} = \frac{(\mathcal{P}_0 - \mathcal{P}_L)\, B^2}{2\, \mu_0\, L} \left(1 - \left(\frac{x_0}{B}\right)^2\right) - \frac{\tau_0\, B}{\mu_0} \left(1 - \frac{x_0}{B}\right) \qquad \text{for } x < x_0 \ . \qquad [8]$$

Q for the whole slit is twice the flow through the half defined by $x \geq 0$:

$$Q = W\, 2 \int_0^B v_z(x)\, dx \quad . \qquad [9]$$

Evaluating the equation for Q is easiest if one uses the trick in BSL Example 2.3-2 and integrates by parts. The result is

$$Q = \frac{2}{3} \frac{W\, (\mathcal{P}_0 - \mathcal{P}_L)\, B^3}{\mu_0\, L} \left(1 - \frac{3}{2}\left(\frac{\tau_0}{\tau_B}\right) + \frac{1}{2}\left(\frac{\tau_0}{\tau_B}\right)^3\right) \text{ for } \tau_B \geq \tau_0 \qquad [10]$$

$$Q = 0 \quad \text{for } \tau_B \leq \tau_0 \qquad [11]$$

with

$$\tau_B \equiv \frac{(\mathcal{P}_0 - \mathcal{P}_L)}{L}\, B \ . \qquad [12]$$

2B.4-2 and
BSLK ~~2B~~.~~2B~~. (c) on p. 71 3.13
(BSLK 2.3-19 and 21 for tube)

Note the similarity in the form of Eqs. [7], [8], and [10] to BSL Eqs. 2.3-25, 26 and 30. Note also that all of these equation revert to the Newtonian equations if $\tau_0 = 0$. In Eq. [10], the first part of the equation matches the Newtonian equation (with μ_0 substituted for μ) while the bracketed term reduces Q below the value for a Newtonian fluid.

c. power-law fluid

The derivation of $v_z(x)$ for a power-law fluid in a slit is given as a homework assignment. The final result is

$$v_z = \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{m\ L}\right)^{1/n} \frac{B^{[1+(1/n)]}}{1+(1/n)} \left(1 - \left(\frac{x}{B}\right)^{[1+(1/n)]}\right) \qquad \text{for } x > 0. \qquad [13]$$

The total flow rate is derived by integrating v_z over x:

$$Q = W\ 2 \int_0^B v_z(x)\, dx \qquad [14]$$

$$Q = \left(\frac{2\ W\ n}{2n + 1}\right) \left(\frac{(\mathcal{P}_0 - \mathcal{P}_L)\ B^{2n+1}}{m\ L}\right)^{1/n} . \qquad [15]$$

Note that for flow of a power-law fluid through a slit, as for power-law flow through a tube, $Q \sim \Delta\mathcal{P}^{1/n}$. Note also that Eq. [15] reverts to the Newtonian formula if $n = 1$ and $m = \mu$.

4. annulus

a. Newtonian fluid - BSL sect. 2.4 (BSLK 2.4)

Notes:

- the basic geometry (i.e., cylindrical) and the momentum balance are same as for flow through circular tube
- BC differ:
  - $v_z = 0$ at $r = R$
  - $v_z = 0$ at $r = \kappa R$

  ($r = 0$ is not within system; therefore can't apply BC that τ_{rz} is finite at $r = 0$)
- BSL defines z pointing *up* this time, unlike sect. 2.3 - alters definition of $\mathcal{P}$

The math gets hairy. Don't let the math distract you from the following:

3.13

An alternate derivation for $v_z(r)$:

The derivation of $v_z(r)$ for Newtonian flow in an annulus in BSL section 2.4 makes use of a clever change of variable in equation 2.4-3. Most students (and probably most professors) would not think of making this change. It is not necessary for solving this problem. You should be able to solve this problem by the straight-ahead method illustrated here. The equation numbers here, after 2.4-2, have no particular relation to the equation numbers in section 2.4.

Start with equation 2.4-2:

$$\tau_{rz} = \left(\frac{P_0 - P_L}{2L}\right) r + \frac{C_1}{r} \tag{2.4-2}$$

Equate τ_{rz} from Newton's law of viscosity with τ_{rz} from the momentum balance:

$$\tau_{rz} = \left(\frac{P_0 - P_L}{2L}\right) r + \frac{C_1}{r} = -\mu \frac{dv_z}{dr} \tag{2.4-3a}$$

$$\frac{dv_z}{dr} = -\left(\frac{P_0 - P_L}{2\mu L}\right) r - \frac{C_1}{\mu r} \tag{2.4-4a}$$

$$v_z = -\left(\frac{P_0 - P_L}{2\mu L}\right) \frac{r^2}{2} - \frac{C_1}{\mu} \ln r + C_2 \tag{2.4-5a}$$

B.C.: $v_z = 0$ at $r = R$ (2.4-6a)

$$0 = -\left(\frac{P_0 - P_L}{2\mu L}\right) \frac{R^2}{2} - \frac{C_1}{\mu} \ln R + C_2 \tag{2.4-7a}$$

$$C_2 = \left(\frac{P_0 - P_L}{2\mu L}\right) \frac{R^2}{2} + \frac{C_1}{\mu} \ln R \tag{2.4-8a}$$

inserting this into Eq. 2.4-7a gives

$$v_z = \left(\frac{P_0 - P_L}{2\mu L}\right) \frac{R^2}{2} \left(1 - \left(\frac{r}{R}\right)^2\right) + \frac{C_1}{\mu} \ln\left(\frac{R}{r}\right) \tag{2.4-9a}$$

B.C.: $v_z = 0$ at $r = \kappa R$ (2.4-10a)

$$0 = \left(\frac{P_0 - P_L}{2\mu L}\right) \frac{R^2}{2} \left(1 - \left(\frac{\kappa R}{R}\right)^2\right) + \frac{C_1}{\mu} \ln\left(\frac{R}{\kappa R}\right) \tag{2.4-11a}$$

$$\frac{C_1}{\mu} \ln\left(\frac{1}{\kappa}\right) = -\left(\frac{P_0 - P_L}{2\mu L}\right) \frac{R^2}{2} \left(1 - \kappa^2\right) \tag{2.4-12a}$$

$$C_1 = -\frac{\mu}{\ln\left(\frac{1}{\kappa}\right)} \left(\frac{P_0 - P_L}{2\mu L}\right) \frac{R^2}{2} \left(1 - \kappa^2\right) \tag{2.4-13a}$$

Plug this expression for C_1 into Eq. 2.4-9a:

$$v_z = \left(\frac{P_0 - P_L}{2\mu L}\right) \frac{R^2}{2} \left(1 - \left(\frac{r}{R}\right)^2\right) - \left(\frac{\mu}{\ln\left(\frac{1}{\kappa}\right)} \left(\frac{P_0 - P_L}{2\mu L}\right) \frac{R^2}{2} \left(1 - \kappa^2\right)\right) \frac{1}{\mu} \ln\left(\frac{R}{r}\right) \tag{2.4-14a}$$

Group terms together with common factor:

3.14

$$v_z = \left(\frac{P_0 - P_L}{4\mu L}\right) R^2 \left(1 - \left(\frac{r}{R}\right)^2\right) - \left(\frac{P_0 - P_L}{4\mu L}\right) R^2 \ln\left(\frac{R}{r}\right) \frac{1}{\ln\left(\frac{1}{\kappa}\right)} \left(1 - \kappa^2\right) \qquad (2.4\text{-}15a)$$

$$v_z = \left(\frac{(P_0 - P_L) R^2}{4\mu L}\right) \left(1 - \left(\frac{r}{R}\right)^2 - \frac{(1 - \kappa^2)}{\ln\left(\frac{1}{\kappa}\right)} \ln\left(\frac{R}{r}\right)\right) \qquad (2.4\text{-}17a)$$

This is Eq. 2.4-14 of BSL.

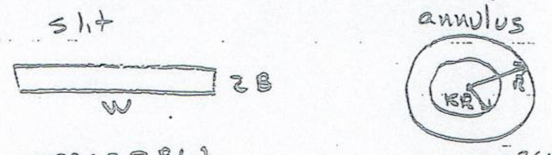
III.D.4.b.

Bingham + Power-Law Fluids in Annulus

Math gets <u>really</u> hairy...

Instead, recall annulus → slit as $\kappa \to 1$;

∴ define "equivalent slit" and use slit eqs. for annulus



area = 2BW
WIDTH = 2B

area = $\pi R^2(1-\kappa^2)$
width = $R(1-\kappa)$

"Equivalent" slit has same area and width as annulus.

Use slit formulas substituting $R(1-\kappa)$ for $2B$ and $\pi R^2(1-\kappa^2)$ for $2BW$

For power-law fluid

$$Q = \pi R^2(1-\kappa^2)\left[\frac{\Delta P\,[R(1-\kappa)]^{n+1}}{2^{n+1}\, m\, L}\right]^{1/n} \frac{n}{2n+1}$$

For Bingham plastic,

let $\tau_B \equiv \left|\frac{\Delta P}{L}\,\frac{R(1-\kappa)}{2}\right|$

if $\tau_B \leq \tau_0$, $Q = 0$

$\tau_B > \tau_0$ $\quad Q = \dfrac{\pi \Delta P R^4 (1-\kappa)^2(1-\kappa^2)}{12\,\mu_0 L}\left[1 - \frac{3}{2}\left(\frac{\tau_0}{\tau_B}\right) + \frac{1}{2}\left(\frac{\tau_0}{\tau_B}\right)^3\right]$

Formulas are "reasonably" accurate for $\kappa > 0.3$.

3.16
3.19

4b ~~5~~. **adjacent flow of immiscible fluids** - BSL sect. 2.5
also BSLK Sect. 2.5

Important points in derivation:

- perform a separate momentum balance on each fluid I and II
- need 2 B.C. on each fluid - total of 4
  - $v_z = 0$ at $x = b$
  - $v_z = 0$ at $x = -b$
  - $v_z^I = v_z^{II}$ at $x = 0$
  - $\tau_{xz}^I = \tau_{xz}^{II}$ at $x = 0$
- Note general form of solution; don't sweat the math details
- Application: "coupling" of flow of wetting and non-wetting fluid in pores, and resulting cross-terms in relative-permeability functions

i. An aside: Definition of $\Delta\mathcal{P}$

$\Delta\mathcal{P}$ is always $(p_0 - p_L) + \rho\, g\, (\Delta \text{ vertical position})$

$$\Delta\mathcal{P} \equiv (p_0 - p_L) + \rho\, g\, [(\text{vertical position})_0 - (\text{vertical position})_L]$$

If z axis is defined as pointing *down*, as in BSL sect. 2.3, then

$$\mathcal{P} \equiv p - \rho\, g\, z$$

$$\Delta\mathcal{P} \equiv (p_0 - p_L) - \rho\, g\, (z_0 - z_L) = (p_0 - p_L) + \rho\, g\, L$$

(same formula applies to Ex. 2.3-2 (p. 48) because z points down there, too)

If z points *up*, as in BSL sect. 2.4, then

$$\mathcal{P} \equiv p + \rho\, g\, z$$

$$\Delta\mathcal{P} \equiv (p_0 - p_L) + \rho\, g\, [(\text{vertical position})_0 - (\text{vertical position})_L] = (p_0 - p_L) - \rho\, g\, L$$

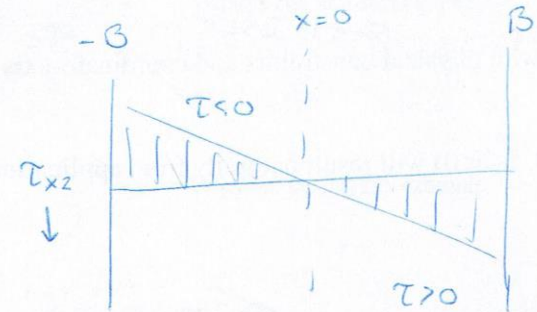
Better than memorizing either formula is to realize the physical significance of the hydrostatic forces; they should *add* to ΔP if flow is downwards and *reduce* ΔP if flow is upwards, against gravity.

recommended procedure:
- assume direction of flow
- calculate pressure at inlet and outlet
- calculate Δp
- Ask: is inlet higher or lower than outlet
  - add or subtract $\rho g(\Delta \text{height})$
- If calculated $\Delta\mathcal{P} < 0$, flow is opposite to what you assumed

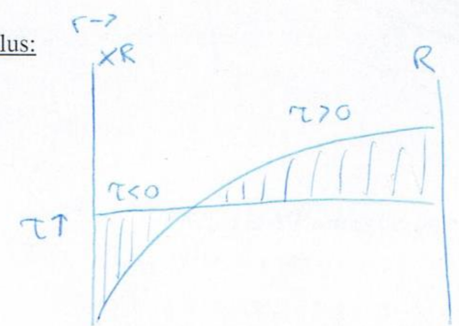
ii. An aside: physical meaning of $\tau < 0$

Now we have had two problems here $\tau < 0$ somewhere:

slit:



annulus:



$r\tau<0$

physical significance: either

- momentum in positive z direction is carried in direction of *decreasing* x or r (both cases here), or
- momentum in the *negative* z direction is carried in the direction of *increasing* x or r

. . . how to handle this mathematically ... ? Which terms represent transport "into" and "out of" the control volume?

iii. An aside: What if fluxes or velocities are *negative*?

(For instance, what if wall is moving in negative z direction; or positive z momentum is transported radially inward (in direction of decreasing r, i.e., $\tau_{rz} < 0$))

See BSLK p.45 middle of page

Principles:

1. Coordinate axes define directions of positive velocity, fluxes.
2. For purposes deriving shell balance, assume all fluxes and velocities are in positive directions, as defined by coordinate axes. (cf. p.43 BSL 2 50-51 BSLK
3. Apply boundary conditions consistently with physical constraints and coordinate-axis directions.
4. Negative velocities or fluxes (e.g., $v_z < 0$, $\tau_{rz} < 0$) will result naturally from application of boundary conditions.

(Don't try to out-smart the process.)

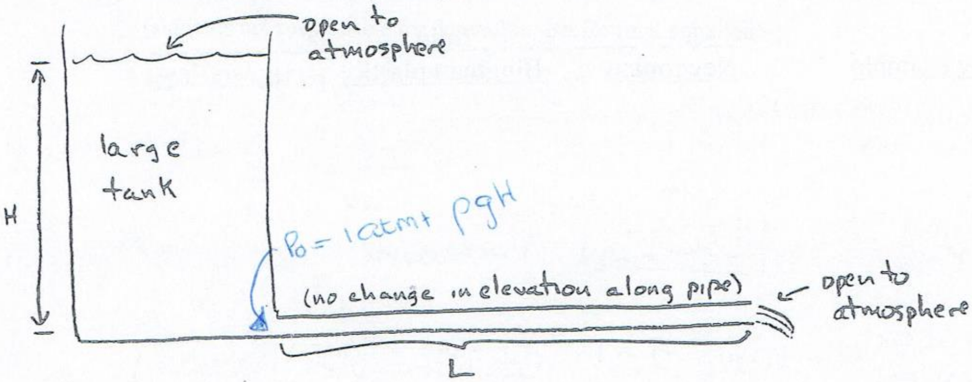
b. non-Newtonian fluid flow in an annulus - on next page

3.18

3.19

5. Correctly including hydrostatic pressure in flow equations

Consider the following example

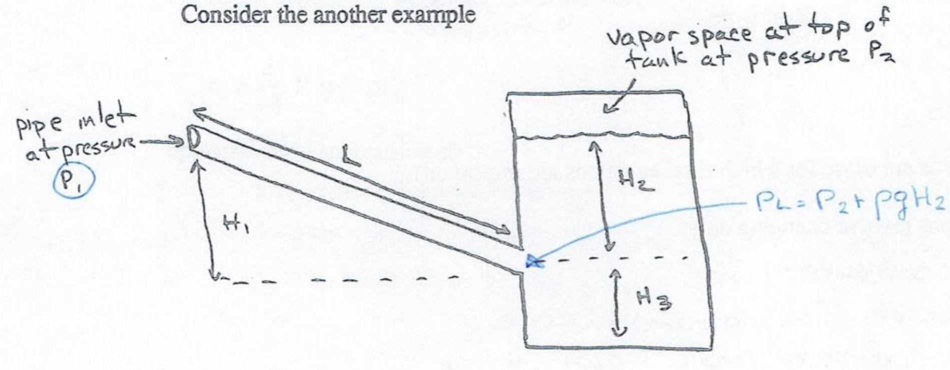


What is $\Delta \mathcal{P}$ across the pipe?

$$\Delta \mathcal{P} = \mathcal{P}_0 - \mathcal{P}_L = [1atm + \rho g H - 1atm] + \rho g (0) = \rho g H$$

$$\frac{\Delta \mathcal{P}}{L} = \frac{\rho g H}{L}$$

Consider the another example



What is $\Delta \mathcal{P}$ across the pipe?

$$\Delta \mathcal{P} = [P_1 - (P_2 + \rho g H_2)] + \rho g H_1 = (P_1 - P_2) + \rho g (H_1 - H_2)$$

$$\frac{\Delta \mathcal{P}}{L} = \frac{(P_1 - P_2) + \rho g (H_1 - H_2)}{L}$$

Moral: include hydrostatic contributions to inlet and outlet pressures in $\Delta \mathcal{P}$

3.20

6. **equation of motion and Navier-Stokes equation** - BSL sect. 3.2
Not required for this course, but used extensively in more-advanced courses in fluid mechanics

7. **recap of examples**

flow example	fluid: Newtonian	fluid: Bingham plastic	fluid: power-law
(parallel plates)			
falling film	$\boxed{\text{BSL 2.2}}$	Homework	
Circular tube	$\boxed{\text{BSL 2.3}}$	$\boxed{\text{BSL Examples}}$	$\boxed{\text{lecture notes}}$
Slit	$\boxed{\text{Homework}}$	$\boxed{\text{Lecture notes}}$	$\boxed{\text{lecture notes homework}}$
annulus	$\boxed{\text{BSL 2.4}}$	$\boxed{\text{lecture notes}}$	$\boxed{\text{Lecture notes}}$
immiscible Fluids	BSL 2.5	homework	

(boxed cases are those for which final equations are important)

all variations involve changing only:

- coordinate axes
- terms in momentum balance
- relation between τ and dv/dx
- boundary conditions

III. E. "Creeping flow" around sphere

1. Newtonian fluid - BSL Sect. 2.6 (BSLK 2.7)

(students not responsible for derivation, but for final equation)

applications of Eq. 2.6-~~16~~17: BSLK eq. 2.7-17

viscosity of fluid

$$\mu = \frac{2}{9} \frac{R^2 (\rho_s - \rho) g}{V_t}$$

radius of sphere

density of ~~fluid~~ sphere

density of fluid

V_t - velocity of sphere relative to fluid (>0 if down)

$$V_t = \frac{\frac{2}{9} R^2 (\rho_s - \rho) g}{\mu}$$

2. suspension of particles in Bingham plastic

students not responsible for derivation, but for final equation:

Particle is completely suspended, as in a solid, if

$$\tau_0 \geq \frac{4}{3\pi} R \, |\rho_s - \rho| \, g$$

applications to suspension of:

- transport of drill cuttings
- transport of "proppant" in hydraulic fracturing

III. F. Assumptions Behind Derivations in BSL Ch. 2

- "slow," rectilinear flow (except flow around sphere)
- incompressible (ρ constant), Newtonian (μ constant) fluid
- steady state ($v(\underline{x})$ independent of t)
- ignore entrance and exit effects
- no slip at walls (v = wall velocity at wall; "BC type 1")
- fluid = continuum

Assumptions break down when flow too fast: "turbulence"
Quantify conditions for breakdown in "Reynolds number:"

$$\mathrm{Re} \equiv \frac{\text{(length)(velocity)(density)}}{\text{viscosity}} \sim \frac{\text{inertial effects}}{\text{viscous effects}} \sim \frac{\rho V^2}{(\mu V/L)}$$

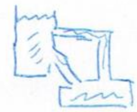
Type of Flow	Def. of Re	Trans. to turbulence	Reference
Horizontal, infinite plates (BSL Fig. 1.1-1)	$\frac{YV\rho}{\mu}$	3000	Schlichting, *Boundary Layer Theory*, 1979, p. 590
Falling Film	$\frac{4\delta\langle v_z\rangle\rho}{\mu}$	20	BSL, 2 p. 46 BSLK p. 48
Slit	$\frac{4B\langle v_z\rangle\rho}{\mu}$	2300	White, *Fluid Mechanics* 1979, p. 433
Circular Tube	$\frac{D\langle v_z\rangle\rho}{\mu}$	2100	BSL, 2 p. 52 BSLK p. 54
Annulus	$\frac{2(R-\kappa R)\langle v_z\rangle\rho}{\mu}$	2000	BSL, 2 p. 56 BSLK p. 60
Flow Around Sphere	$\frac{D_{sph}V_{fl}\rho_{fl}}{\mu_{fl}}$	0.1 *	BSL, 2 pp. 61. BSLK p. 68

* computed V_{fl} is within about 10% of true value for $\mathrm{Re} \leq 1$

These values apply only to Newtonian Fluids!

What about flow through multiple tubes in series or in parallel

series



3.23

III. G. Analogies to electrical conduction

1. Linear relationship between Q and ΔP/L

In electrical conduction, I (current) is proportional to voltage difference ∇V

In laminar flow of Newtonian fluids, Q is proportional to ΔP .

This linear relationship between driving force and transport rate does not apply to non-Newtonian fluids, nor to turbulent flow of Newtonian fluids.

2. Multiple elements in flow path

In both electrical conduction and flow of fluids, the transport may be more complicated than just through one channel. In electrical conduction, if the current passes through more than one resistor in series , then

Current is same in all resistors
One adds voltage differences to get the total voltage difference

In electrical conduction, if two resistors are in parallel , then

the voltage is the same in both
one adds current in the two to get the total current

In flow of fluids through multiple channels in series

Q (or W) is same for all channels
One adds ΔP.

In flow of fluids through multiple channels in parallel

ΔP is same for all channels
One adds flow rates Q's.

This analysis of flow through multiple channels applies to laminar or turbulent flow, in tubes, slits, annuli, or porous media, and to Newtonian and non-Newtonian fluids.