IX. Friction Factors

A. Yet another review (!)

Each transport problem involves:

	"balance" conservation equation	transport law
laminar flow	Shell momentum balance	constitutive eq. relating τ to dv/dx
turbulent flow	macroscopic momentum balance → friction ~~ratio~~ factor	correlation ~~betwe~~ for f as function of Re

B. General definition of f

$$F_k = AKf \; ; \; f = \frac{F_k}{AK} \quad (I)$$

applicable for: • flow through cylindrical tube • spherical object falling/rising through fluid • reactor packed with sediment

$F_k \equiv$ drag force on solid

A ≡ "characteristic area" of solid

K ≡ kinetic energy/unit volume of fluid

C. Tube flow (BSL Sect. 6.2) (Also BSLK)

1. Definitions

$A \equiv$ wetted surface $= 2\pi RL$

$K \equiv (1/2)\, \rho \langle v\rangle^2$

$\therefore\; F_k \equiv (2\pi RL)(\frac{1}{2}\rho \langle v\rangle^2) f \quad (II)$

(eq 6.1-2 BSLK)

2. Macroscopic momentum balance on fluid in tube

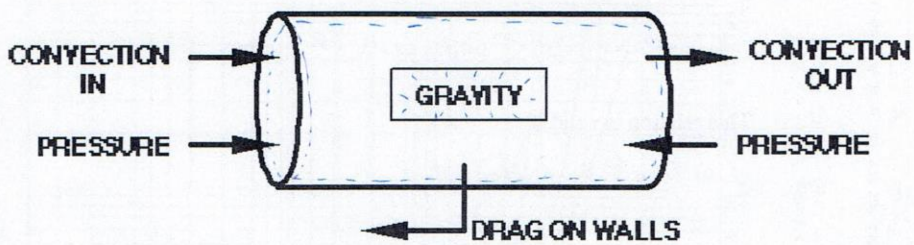
Recall can use macroscopic balance only if either

1) In turbulence, $\langle v \rangle$ is nearly uniform across the tube

or

2) we don't care how V_z varies with r. Only interested in averge, $\langle v_z \rangle$

For turbulent flow, condition (1) applies reasonably well. More fundamentally, condition (2) applies: we don't care how v_z varies across tube; we only want to know drag as function of average v_z. It's understood in what follows that "v" means average velocity, $\langle v_z \rangle$.))



Define system as interior of pipe.

Terms in momentum balance:

convection in: $\underbrace{(\pi R^2)}_{\text{area}} [\rho v^2|_{Z=0}]$

convection out: $(\pi R^2) [\rho v^2|_{Z=L}]$

(for incompressible fluids, these terms cancel)

pressure at Z = 0: $(\pi R^2)\, p_0$

pressure at Z = L: $(\pi R^2)\, (-p_L)$

(negative because pressure at Z=L acts in negative z direction)

body force (gravity): $\underbrace{(\pi R^2 L)}_{\text{volume}} \rho g \cos \beta$

(combine gravity and pressure into $(\pi R^2)(\mathcal{P}_0 - \mathcal{P}_L)$

$\equiv (\pi R^2)\, \Delta\mathcal{P}$

drag force on walls: $- F_k$ $(2\pi R L \tau|_{r=R})$

(negative because F_k is defined as positive if momentum leaves system)

No accumulation (acceleration) at steady state

__Momentum balance__:

$\Delta\mathcal{P}\, \pi R^2 - F_k = 0$; or $F_k = \Delta\mathcal{P}\, \pi D^2 / 4$ (III)

$F_k = A \cdot K \cdot f \qquad + F_k = \frac{\Delta P \pi D^2}{4}$

9.3

combine two equations for F_k from sections 1 and 2 above -->

$$f = \frac{1}{4}\left(\frac{D}{L}\right)\frac{\Delta P}{\frac{1}{2}\rho\langle v\rangle^2}$$ (BSL K eq. 6.1-4)

3. Correlations for f

dimensional analysis says f = f(Re, tube shape)

"tube shape" means

- cross-sectional shape
- roughness of pipe wall (K)

This relation is valid if

a) far from entrance

or

b) $L/D >> 1$

Basic correlation is BSL Fig. 6.2-2

this chart represents transport Law

for turbulent flow of Newtonian fluids in pipes;

Eq. 6.1-4 represents the conservation equation or 'balance'

a. for Re < 2100

$f = \frac{16}{Re}$ (Fanning friction factor); Tube roughness is not important.
The chart (Fig. 6.2-2) is only valid for cylindrical pipe shapes

b. for Re > 2100

for circular tubes:

f depends on . . . Reynolds number (Re) and K/D, where
K = 'height of protuberance' . ΔP greatly increased
over laminar flow roughness

from Perry + Chilton, "Chemical Engineers' Handbook, 5th Edition

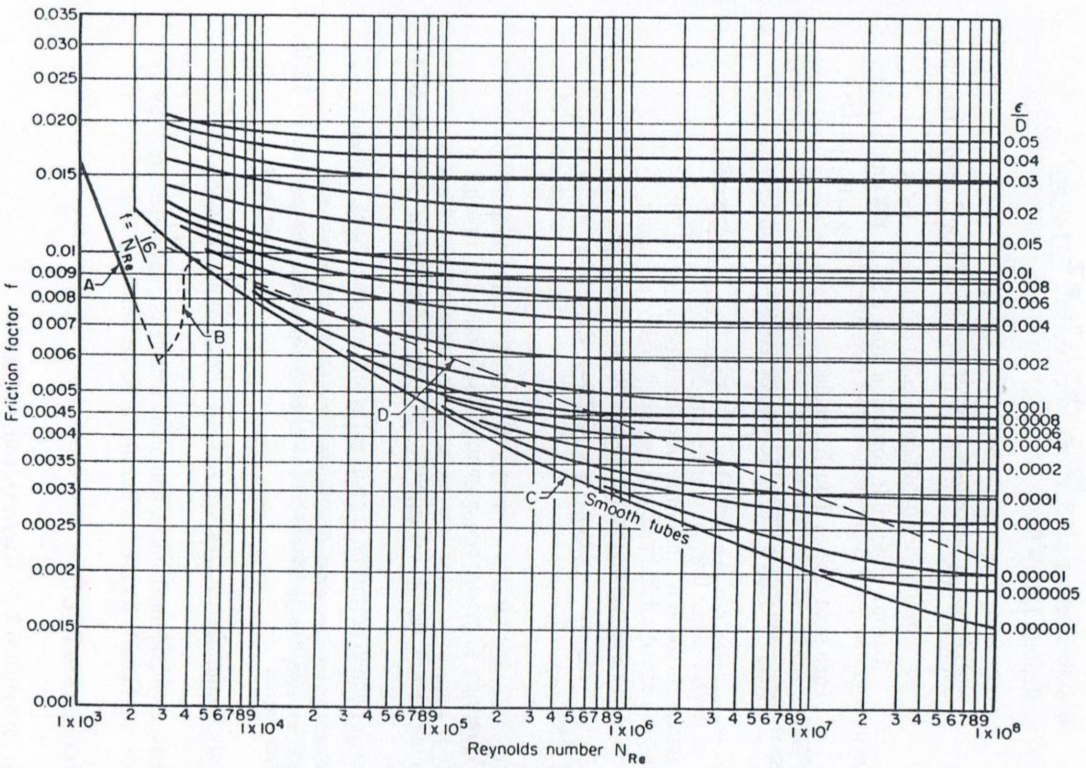


Fig. 5-26. Fanning friction factors. Reynolds number $N_{Re} = DV\rho/\mu$, where D = pipe diameter, ft.; V = velocity, ft./sec.; ρ = fluid density, lb./cu. ft.; μ = fluid viscosity, lb./(ft.)(sec.) = cp./1488. [*Based on Moody, Trans. Am. Soc. Mech. Engrs.*, **66**, 671 (1944).]

For rough estimates or checks, the velocity-head concept [Lapple, *Chem. Eng.*, **56**(5), 96–104 (1949)] can be applied to the first two forms of Eq. (5-52). The velocity head is $V^2/2g_c = h_v$ and the number of velocity-head losses in straight pipe is $4fL/D$. Typical values of h_v and L/D for 1 velocity-head loss are given in Table 5-9.

For cross sections other than circular of ducts running full or for open channels when the variation in depth is negligible, where

Table 5-10. Coordinates for Liquids and Aqueous Solutions

For use with Fig. 5-27

	X	Y		X	Y
Acetaldehyde	−0.3	3.7	Glycerol, 100%	6.9	1.8
Acetic acid, 100%	1.0	4.0	Glycerol, 50%	3.0	3.7
Acetic acid, 77%	2.6	3.8	Hydrochloric acid, 31.5%	1.1	4.2
Acetic anhydride	0.7	4.3	Linseed oil, raw	3.4	1.8
Acetone, 100%	0.9	3.4	Mercury	See chart	
Acetone, 35%	2.7	3.7			

Turbulent reg

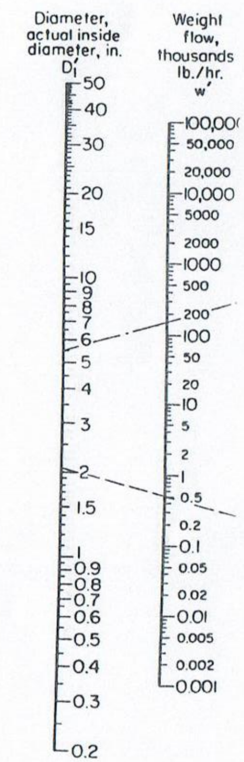


Fig. 5-27. Pipe-flow chart. [*Genereau*

the flow is in the turbulent-flow re
Fanning equation, Eq. (5-52), are
occurs, is replaced by the hydrauli

d. Noncircular tubes: "hydraulic radius approximation"

Define $R_h \equiv S/Z$ (BSLK Eq. 6.2-16)

$S \equiv$ cross-sectional area

$Z \equiv$ perimeter of cross-section

$D_h = 4R_h = 4 \cdot \frac{S}{Z}$

Replace D by 4 R_h

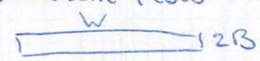
Compute $\langle v \rangle = \frac{Q}{S}$ with S for <u>true cross-sectional shape</u> (not circle)

beware of possible confusion from BSL2 p. ~~188~~ 183 on this point BSLK p. 168

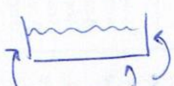
Then treat tube as though circular

<u>NOTE</u>: valid only for turbulent flow

example: narrow slit



another example: canal



wetted perimiter:

i. For rectangular slit or natural fracture,

$R_H = \frac{2B \cdot W}{2W+4B}$ where $2B \approx 0 \rightarrow R_H = B$ and $D_H = 4R_H$

4. Warning: There are two common "friction factors" in use

- "Fanning friction factor" differs from "Moody friction factor" by factor of 4
- Beware of which definition is assumed in any reference book you use
- BSL uses "Fanning friction factor", and so do we in this course; has f = 16/Re in laminar region
- "Moody friction factor" charts have f = 64/Re in laminar region
    - most civil and mechanical, and many petroleum, engineers use "Moody friction factor"; it's used in Petroleum Engineers handbook

5. examples in BSL

a. example 6.2-1: Given Q, compute ΔP/L

9.7

b. example 6.2-2. Given $\Delta P/L$, compute Q.

Problem: Re is function of unknown <v>

<u>Method 0</u>: trial and error (not in BSL; (1,2 or K) ~~taught in CE 319~~)

- guess <v>
- compute Re given <v> from definition of Re
- determine f(Re) from Fig. 6.2-2 (transport law)
- compute <v> from f using eq. 6.1-4 (conservation eq.)
- recompute Re, f, <v>, etc., until $\exists$ no further changes; at this point solution satisfies both conservation equation and transport law

Warning:
The particular trial-and-error numerical method given above as "Method 0" is called "successive substitution." It is not in general an effective numerical method, though it works OK for friction-factor problems.
Don't worry about solution methods in BSL.

6. **final notes**

Note that equation 6.2-12 ("Blasius eq."), also written on Fig. 6.2-2, is valid only for extremely smooth tubes for 2100 < Re < 100,000. This can be confusing because this same equation is written on Fig. 6.2-2. To repeat, *it is valid only for hydraulically smooth tubes*.

In fact, Eqs. 6.2-12 to 6.2-~~15~~ 14 all apply only under limited conditions. [redacted]

Don't sweat theoretical discussion of this equation in middle of BSL2 pp. 180-81. (BSLK 165-166

Example 6.2-2 reworked by trial-and-error method.

Determine the flow rate, in kilograms per hour, of water at 20°C through a 305 m length of horizontal 20.32 cm pipe (internal diameter 20.27 cm) under a differential pressure of 20.68 kPa. For such a pipe use Fig. 6.2-2 and assume that $k/D = 2.3 \times 10^{-4}$.

Method ①

guess $\langle v \rangle$

determine Re $\left(Re = \dfrac{D \langle v \rangle \rho}{\mu} \right)$

Find f using Fig 6.2-2

Transport law

compute $\langle v \rangle$ from eq. 6.1-4

Conservation Eq.

$$f = \frac{1}{4}\left(\frac{D}{L}\right)\left(\frac{P_0 - P_L}{\frac{1}{2}\rho \langle v \rangle^2}\right)$$

rearranging we get

$$\langle v \rangle = \left[\frac{1}{4}\left(\frac{D}{L}\right) \frac{P_0 - P_L}{\frac{1}{2}\rho f} \right]^{1/2}$$

9.10

1) guess $\langle v \rangle = 0.05$ m/s

$$Re = \frac{D \langle v \rangle \rho}{\mu} = \frac{(0.2027\,m)(0.05\ m/s)(1000\,Kg/m^3)}{0.00103\ Pa\cdot s}$$

$$= 1.97 \cdot 10^5 \langle v \rangle = 9850$$

$f \approx 0.0077$ (from Fig 6.2-2)

2) calculate $\langle v \rangle = \left[\frac{1}{4}\left(\frac{D}{L}\right)\frac{(\mathcal{P}_0 - \mathcal{P}_L)}{(1/2)\rho f}\right]^{1/2} = \left[\frac{1}{4}\frac{0.2027\,m}{305\,m}\left(\frac{20680\,Pa}{1/2\,(1000\,\frac{Kg}{m^3}\,f)}\right)\right]^{1/2}$

$$= \left(6.87 \cdot 10^{-3}/f\right)^{1/2} = 0.942\ m/s$$

$\rightarrow Re = 1.86 \cdot 10^5 \rightarrow f = 0.0045 \rightarrow \langle v \rangle = 1.23\,m/s$

$\longrightarrow \quad = 2.43 \cdot 10^5 \qquad = 0.0043 \rightarrow \qquad = 1.26$

$= 2.47 \cdot 10^5 \qquad = 0.0043 \rightarrow \langle v \rangle = 1.26$

{Note $Re = 2.47 \cdot 10^5$ is approximately the same result as in BSLK Fig 6.2-3 B

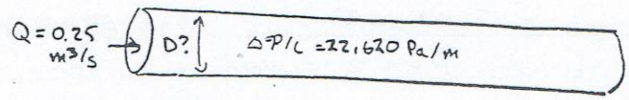
BSLK velocity = 4.01 ft/s = 1.22 m/s. (p. 171)

same as our answer to within ability to read chart

9.11

Examples of trial-and-error method

To put out the oilfield fires in Kuwait, firefighters used hoses that could deliver up to 4000 gal/min (0.25 m^3/s) of seawater. The hoses must extend long distances, so suppose $\Delta P/L$ is limited to 1 psi/ft (22,620 Pa/m). The density and viscosity of water are roughly 1,000 kg/m^3 and 0.001 Pa s, respectively. What hose diameter D is required? Assume the roughness factor k/D is 0.004. Note the figure given at the end of the exam.
(25 points)



In this problem, we don't know Re since we don't know D. There are at least 2 ways to solve this problem.

1) Trial + error. $f = \frac{1}{4}\left(\frac{D}{L}\right)\left(\frac{\Delta P}{\frac{1}{2}\rho\langle v\rangle^2}\right)$. Since Q is fixed, not $\langle v\rangle$, $\langle v\rangle = Q/[(\pi/4)D^2]$

$$\rightarrow f = \frac{1}{4}\frac{D}{L}\frac{\Delta P(\frac{\pi}{4}D^2)^2}{\frac{1}{2}\rho Q^2} = \frac{1}{4}\frac{\Delta P}{L}\frac{(\pi/4)^2 D^5}{\frac{1}{2}\rho Q^2} = \frac{1}{4}(22,620)\frac{(\pi/4)^2 \quad D^5}{(\frac{1}{2})(1000)(0.25)^2} = 111.6\ D^5 \quad \boxed{I}$$

$$Re = \frac{D\langle v\rangle\rho}{\mu} = \frac{DQ\rho}{\mu\frac{\pi}{4}D^2} = \frac{Q\rho 4}{\mu\pi D} = \frac{(0.25)(1000)(4)}{(0.001)\pi D} = 3.18\cdot 10^5/D \quad \boxed{II}$$

Guess $D = 0.1\,m \rightarrow Re = 3.2\cdot 10^6$; from chart, $f = 0.0072 = 111.6\ D^5 \rightarrow D = 0.145\,m$

Guess $D = 0.145\,m \rightarrow Re = 2.19\cdot 10^6$; from chart, $f = 0.0072$ again. $\rightarrow D = 0.145\,m$

done. $D = 0.145\,m$ (about $5\frac{3}{4}$ in.)

Stonewall has been assigned to design a large pipeline to transport 0.2 m^3/s oil products with a pressure gradient of 10 psi/mile (42.8 Pa/m). The pipeline is horizontal. The oil products have density 850 kg/m^3 and viscosity 10 cp (0.01 Pa s). Stonewall figures that even without knowing the pipe diameter, a reasonable estimate of roughness is k/D = 0.01. What pipe diameter should he recommend?

Without knowing Re, one can't proceed directly. With D unknown, we don't know v or D. $v = Q/(\pi D^2/4) = 0.2/(\pi D^2/4) = 0.255/D^2$.

$Re = Dv\rho/\mu = D[0.255/D^2]\,850/(0.01) = 21645/D$

$f = \frac{1}{4}\frac{D}{L}\frac{\Delta P}{\frac{1}{2}\rho v^2} = (1/4)D(42.8)/[\frac{1}{2}(850)(0.255/D^2)^2] = 0.387\,D^5$ (eq. 6.1-4)

From here, one can proceed in several ways.

1) Trial + error. Guess $D = 1\,m$, $Re = 21,645$. From f(Re), $f \approx 0.0105 = 0.387D^5$

$\rightarrow D = 0.486 \rightarrow Re = 44,500 \rightarrow f(Re) = 0.0099 = 0.387D^5 \rightarrow D = 0.480$

$\rightarrow Re = 45100 \rightarrow f \approx 0.0099$ again. Done. $D = 0.48\,m$.

9.12

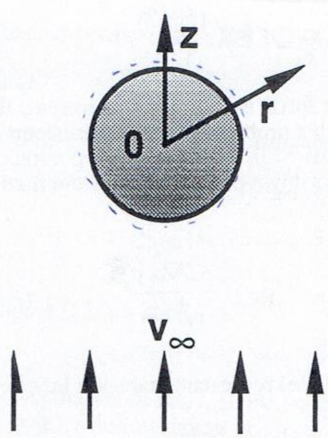
D. flow around spheres (BSL Sect. 6.3)

1. Definitions

BSLk 6.3-14

$$F_k = A K f \equiv (\pi R^2)(\tfrac{1}{2}\rho V_\infty^2) f \qquad \text{(I)}$$

2. macroscopic momentum balance



Note coordinate system is centered on sphere, and z points up; fluid flows up past sphere, which is stationary in this coordinate system.

Define system as solid sphere.

<u>Terms in momentum balance</u>:

drag on sphere: F_k

(positive, because drag pushes sphere in positive z direction)

gravity: $-\frac{4}{3}\pi R^3 \rho_s g$ (gravity pulls down)

(pulls sphere in negative z direction)

In BSLk

buoyancy: $\frac{4}{3}\pi R^3 \rho g$

(ρ without subscript refers to liquid; buoyancy pushes sphere in positive z direction)

there is no accumulation (because balance is at steady state) and no convection across system boundaries (i.e., into or out of sphere)

<u>Momentum balance</u>:

$$F_k - \frac{4}{3}\pi R^3 (\rho_s - \rho_{fl}) g = 0 \text{ (steady state)} \qquad \text{(II)}$$

$$\hookrightarrow F_k = \frac{4}{3}\pi R^3 (\rho_s - \rho_{fl}) g$$ J BSLk eq. 6.1-6

combine eqs. I, II -->

$$f = \frac{4}{3}\left(\frac{gD}{V_\infty^2}\right)\left[\frac{\rho_s - \rho_{fl}}{\rho_{fl}}\right]$$ BSL K (BSL eq. 6.1-7)

or

$$v_\infty = \sqrt{\frac{4}{3} g D \left(\frac{1}{f}\right)\left(\frac{\rho_s - \rho_{fl}}{\rho_{fl}}\right)}$$

In either form, this equation represents the conservation equation for this problem; still need transport law

If $(\rho_s - \rho) < 0$, then simply drop minus sign, and note that fluid flows <u>down</u> past sphere (sphere floats *up* through fluid)

3. correlation for f

Again, f = f(Re); $\mathrm{Re} \equiv \frac{D V_\infty \rho}{\mu_{fl}}$; (ρ and μ refer to fluid)

correlation for f(Re) represents transport law for this problem;

given by Fig 6.3-1

this correlation is for an isolated, smooth sphere in an infinite body of fluid; roughness and spinning of the sphere can greatly affect its behavior, as can interference from a solid wall or other spheres; for further discussion of spinning and roughness, see R. G. Watts and A. T. Bahill, *Keep Your Eye on the Ball: The Science and Folklore of Baseball*, W. H. Freeman and Co., New York, 1990.

9.15

4. Solving for unknown Re

This is the usual case in problems of falling spheres: unknown is usually D, v_∞, ρ or μ. Can't compute Re without knowing answer to problem

cf. BSL Example 6.3-1: don't know D

Method O: trial and error (not in BSL; taught in CE 319)

- guess D
- compute Re from assumed D using definition of Re
- Determine f from calculated Re using Fig. 6.3-1

(transport law)

- recompute D from eq. 6.1-7

(conservation of momentum)

- repeat calculation of Re, f, D, etc., until $\exists$ no further changes - at this point, solution satisfies both conservation equation and transport law

9.17

BSCK

Example 6.3-1. (BSL) by trial and error method

Given: $\rho = 1.59\ gcm^{-3} = 1590\ kgm^{-3}$
$\rho s = 2.62\ gcm^{-3} = 2620\ kgm^{-3}$
$\mu = 9.58$ millipoise $= 9.58 \times 10^{-4}\ kgm^{-1}s^{-1}$
$T = 20\ ^{o}C$
$V_\infty = 65\ cms^{-1} = 0.65ms^{-1}$
$D = ?$

$$f = \frac{4}{3}\frac{gD}{v_\infty{}^2}\left(\frac{\rho_S - \rho}{\rho}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 1$$

$$\text{Re} = \frac{D\rho V_\infty}{\mu} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 2$$

Solution procedure:

Guess D (arbitrary), Solve for Re, read f from chart (Fig. 6.3-1 BSL page 192), Recalculate D, then repeat calculation of Re, f and D until D equals guessed value.

steps : Substituting values into equations 1 and 2 above gives:

$$f = \frac{4}{3}\frac{9.81}{0.65^2}\left(\frac{2620 - 1590}{1590}\right)\text{D} = 20.055\ \text{D}$$

$$\text{Re} = \frac{\text{D}x1590\ x0.65}{9.58\ x10^{-4}} \approx 1078810\ \ \text{D}$$

Guessing D = 0.04m
Re = 1078810 x 0.04 = 43152.4 and $f \cong 0.48$

Recalculating D twice gives:

D = 0.48 / 20.055 $\cong$ 0.0239m and
Re = 1078810 x 0.0239 = 25820, giving $f \cong 0.44$ from chart.

D = 0.44 / 20.055 $\cong$ 0.0219m and
Re = 1078810 x 0.0214 = 23098, giving $f \cong 0.44$ from chart.

Since the value of f is same, **D = 0.0219m = 2.19cm**

9.18

An example of trial and error in falling-sphere problem

Safety experts warn gun owners that firing into the air can be dangerous, because the bullet can fall to earth with enough velocity to hurt someone. Suppose a bullet were a sphere of lead (density 11,300 kg/m^3) weighing 1/2 ounce (0.0156 kg). What would be the terminal velocity of such a sphere falling from a great height through air (with viscosity $\mu = 1.8 \times 10^{-5}$ Pa s and density $\rho = 1.3$ kg/m^3)?
(20 points)

We don't know v, or Re. ∴ some sort of special or trial-and-error solution is needed. ∃ several possible ways to proceed.

1) trial + error: $Re = Dv\rho/\mu$; what is D? $\frac{4}{3}\pi R^3 \rho = 0.0156$.

$\frac{4}{3}\pi R^3 (11{,}300) = 0.0156 \rightarrow R = 0.00691$ m; $D = 0.0138$ m

$$Re = \frac{(0.0138)v(1.3)}{(1.8 \cdot 10^{5})} = 997\ v$$

$$f = \frac{4}{3} gD \frac{1}{v^2}\left(\frac{\rho_s - \rho}{\rho}\right) = \frac{4}{3}(9.9)(0.0138)\frac{1}{v^2}\left(\frac{11{,}300 - 1.3}{1.3}\right) = 1567/v^2$$

guess $v = 1$ m/s. $Re = 997$. $f \approx 0.45 = 1567/v^2 \rightarrow v = 59.0$ m/s

from Fig 6.3-1

guess $v = 59.0$. $Re = 5.9 \cdot 10^4$. from Fig 6.3-1, $f \cong 0.49$ $v = \sqrt{\frac{1567}{0.49}} = 56.6$

guess $v = 56.5$ $Re = 5.64 \cdot 10^4$. " $f = 0.49$ No change from last iteration; ∴ done. $v = 56$ m/s

order steps:

guess unknown

calculate Re

find f(Re) from chart

calculate unknown from f

did unknown change?

yes:

no: done

E. packed columns.

1. **Applications in Petroleum Engineering** and Groundwater Flow

- Unconsolidated/ ~~likely~~ lightly consolidated geological Formation
- Gravel Pack around well = 'Filter' around well filled with Sand particles
- propped Fractures
  - unless plugged by fine Particles chemical Gunk

not valid for

- Consolidated / cemented geological Formation
- Packing plugged by particles etc.
- Packing of deformable particles

2. **Definitions**

$D_p \equiv$ Particle diameter (if spherical)

(see BSL if particle is not spherical)

$G_0 \equiv \rho V_0$

$v_0 \equiv$ Darcy velocity = $\frac{Q_t}{A}$ -> cross sectional area of pack

(would be written as "u" or Q/A in petroleum engineering)

$\varepsilon \equiv$ Porosity

(would be written "ϕ" in petroleum engineering)

$\rho \equiv$ Density of Fluid if incompressible

(see BSL if fluid is very compressible)

3. correlation for f

Fig. 6.4-2 gives correlation for $\left(f\left(\frac{\varepsilon^3}{1-\varepsilon}\right)\right)$ (vertical axis) v. Re (horizontal axis)

easier to use Ergun equation since it covers entire extent of chart

Note $Re \equiv \left(\frac{D_p\, G_o}{\mu}\right)\left(\frac{1}{1-\varepsilon}\right) \equiv \frac{D_p V_o \rho}{\mu} \cdot \frac{1}{1-\varepsilon}$

(note volumetric flux v_o in definition of Re instead of velocity)

For most applications, it's simpler to just use equations in text:

Don't need to worry about laminar vs. turbulent – eq. 6.4-12 valid for all Re; or

eq. 6.4-9 valid for $Re \leq 10$ and $\varepsilon < 0.5$

eq 6.4-11 valid for $Re \geq 1{,}000$

4. Comparison to Darcy's law - k for sandpack ← as defined by petroleum engineers

For $Re \leq 10$, eq. 6.4-9 applies:

$$v_o \equiv \frac{Q}{A} = \frac{\Delta P}{L} \frac{D_p^2}{150\mu} \cdot \frac{\varepsilon^3}{(1-\varepsilon)^2}$$

'specific discharge' in soil mechanics

compare to Darcy's law, also valid only at low Re (absence of turbulence):

$$\frac{K}{A}\frac{\Delta h}{\Delta L} \qquad \frac{Q}{A} = \frac{k}{\mu}\frac{\Delta P}{L}$$

as used by petroleum engineers

→ $\boxed{K = \frac{D_p^2}{150}\frac{\varepsilon^3}{(1-\varepsilon)^2}\frac{\rho_w g}{\mu_{g1}}}$

$$\rightarrow k = \frac{\cancel{D_p^2}\,\cancel{D_p^2}\; D_p^2}{150}\frac{\varepsilon^3}{(1-\varepsilon)^2} \qquad \text{for } Re \equiv \left(\frac{D_p\, v_o\, \rho}{\mu}\frac{1}{1-\varepsilon}\right) \leq 10$$

permeability

(Note this implies a transition to turbulence at much lower flow rate than implied by BOT model)

Why use these correlations for packed beds instead of the BOT model? BOT model requires measured k and ϕ. Eq. 6.4-9 *predicts* k for given ε and D_p.