DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Thursday July 4 2013, 18:30-21:30

1. We use the Forward Euler Method to integrate the initial value problem defined by $y' = f(t, y)$, $y(t_0) = y_0$.

   [a] Show that the local truncation error of the Forward Euler Method is given by $O(h)$. *It is not allowed to use the test–equation.* (2pt)

   [b] Derive the amplification factor for the Forward Euler Method. (2pt)

   Given the initial value problem

$$\begin{cases} \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = \sin t, \\ y(0) = -1, \quad \dfrac{dy}{dt}(0) = 0. \end{cases} \tag{1}$$

   [c] Show that $y(t) = -cos(t)$ is the solution to this initial value problem. (1pt)

   [d] Rewrite this initial value problem into the form of a system of first–order differential equations. Take the initial conditions into account. (1pt)

   We continue with the following system of initial value problems:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}. \tag{2}$$

   [e] Perform one step with the Forward Euler Method with $h = 0.5$ and $t_0 = 0$ to the above system with initial conditions $x_1(0) = -1$ and $x_2(0) = 0$. (2pt)

   [f] Derive the range for the time–step h where the Euler Forward Method applied to system (2) is stable. (2pt)

[0] **please turn over, For the answers of this test we refer to: http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html**

2. We analyze Lagrangian interpolation. For given points x_0, x_1 …, x_n, with their respective function values $f(x_0)$, $f(x_1)$, …, $f(x_n)$, the interpolatory polynomial $p_n(x)$ is given by

$$
\begin{aligned}
p_n(x) &= \sum_{i=0}^{n} f(x_i) L_i(x), \text{ with} \\
L_i(x) &= \frac{(x-x_0)(\ldots)(x-x_{i-1})(x-x_{i+1})(\ldots)(x-x_n)}{(x_i-x_0)(\ldots)(x_i-x_{i-1})(x_i-x_{i+1})(\ldots)(x_i-x_n)}.
\end{aligned}
\tag{3}
$$

Further, the following measured values have been given in tabular form:

i	x_i	$f(x_i)$
0	0	1
1	1	2
2	2	4

(a) Give the linear Lagrangian interpolatory polynomial with nodes x_0 and x_1. (1pt.)

(b) Give the quadratic Lagrangian interpolatory polynomial with nodes x_0, x_1 and x_2. (2pt.)

(c) Approximate $f(0.5)$ both by using linear and quadratic Lagrangian interpolation. (2pt.)

The Newton-Raphson method is based on the following formula:

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}.$$

(d) Derive the above formula for the Newton-Raphson method. (2pt.)

(e) We are searching the positive zero of $f(x) = x^2 - 2x - 2$. Use $p_0 = 2$ as the initial guess and determine p_1 by the use of the Newton-Raphson method. (1pt.)

(f) Let p be the solution of $f(p) = 0$. Demonstrate that

$$|p - p_{n+1}| = K|p - p_n|^2, \text{ for } n \to \infty \tag{4}$$

and determine the value of K. (2pt.)