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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Thursday July 4 2013, 18:30-21:30

1. [a] The local truncation error is given by

$$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h}, \tag{1}$$

where z_{n+1} is the result of one Forward Euler step starting from y_n. We determine y_{n+1} by the use of a Taylor Series around t_n:

$$y_{n+1} = y_n + hy'(t_n) + O(h^2). \tag{2}$$

We realize that

$$y'(t_n) = f(t_n, y_n). \tag{3}$$

Hence, this gives

$$y_{n+1} = y_n + hf(t_n, y_n) + O(h^2). \tag{4}$$

For z_{n+1}, after substituting y_n into Forward Euler, one obtains

$$z_{n+1} = y_n + hf(t_n, y_n). \tag{5}$$

Then, it follows that

$$y_{n+1} - z_{n+1} = O(h^2), \text{ and hence } \tau_{n+1}(h) = \frac{O(h^2)}{h} = O(h). \tag{6}$$

[b] We use the test–equation $y' = \lambda y$, then it follows that

$$w_{n+1} = w_n + h\lambda w_n = (1 + h\lambda)w_n \tag{7}$$

Hence the amplification factor is given by

$$Q(h\lambda) = 1 + h\lambda. \tag{8}$$

[c] Consider $y(t) = -cos\ t$, then $y'(t) = sin\ t$ and $y''(t) = cos\ t$. Hence

$$y''(t) + y'(t) + y(t) = cos\ t + sin\ t - cos\ t = sin\ t, \tag{9}$$

and hence $y(t) = -cos\ t$ is a solution to the differential equation. Further, $y(0) = -cos\ 0 = -1$ and $y'(0) = sin\ 0 = 0$, and hence the initial conditions are also satisfied.

[d] Let $x_1 = y$ and $x_2 = y'$, then it follows that $y'' = x_2'$, and hence we get

$$\begin{aligned} &x_2' + x_2 + x_1 = \sin(t), \\ &x_2 = x_1'. \end{aligned} \tag{10}$$

This expression is written as

$$\begin{aligned} &x_1' = x_2, \\ &x_2' = -x_1 - x_2 + \sin(t). \end{aligned} \tag{11}$$

Finally, we get the following matrix–form:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(t) \end{pmatrix}. \tag{12}$$

Here, we have $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ and $f = \begin{pmatrix} 0 \\ \sin(t) \end{pmatrix}$. The initial conditions are given by $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

[e] The Forward Euler Method, applied to the system $\underline{x}' = A\underline{x} + \underline{f}$, gives at the first step:

$$\underline{w}_1 = \underline{w}_0 + h\left(A\underline{w}_0 + \underline{f}_0\right). \tag{13}$$

With the initial condition and $h = 0.1$, this gives

$$\underline{w}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \frac{1}{2}\left(\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 0.5 \end{pmatrix}. \tag{14}$$

[f] To this extent, we determine the eigenvalues of the matrix A. Subsequently, these eigenvalues are substituted into the amplification factor. The eigenvalues of A are given by $-\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$. Substitution into the amplification factor gives

$$Q(h\lambda) = 1 + h\lambda = 1 + \frac{h}{2}(-1 + i\sqrt{3}) = 1 - \frac{h}{2} + \frac{h\sqrt{3}}{2}i. \tag{15}$$

Herewith, it follows that

$$|Q(h\lambda)|^2 = (1 - \frac{h}{2})^2 + \frac{3h^2}{4} = 1 - h + h^2. \tag{16}$$

Since, for numerical stability, we need $|Q(h\lambda)| \leq 1$, we get

$$h^2 - h \leq 0 \Longleftrightarrow h \leq 1, \tag{17}$$

and hence for $0 \leq h \leq 1$, we have numerical stability, so $h \in [0, 1]$.

2. (a) The linear Lagrangian interpolatory polynomial, with nodes x_0 and x_1, is given by

$$p_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1). \tag{18}$$

This is evident from application of the given formula.

(b) The quadratic Lagrangian interpolatory polynomial with nodes x_0, x_1 and x_2 is given by

$$p_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2). \tag{19}$$

This is also evident from application of the given formula.

(c) To this extent, we compute $p_1(0.5)$ and $p_2(0.5)$ for both linear and quadratic Lagrangian interpolation as approximations at $x = 0.5$. For linear interpolation, we have

$$p_1(0.5) = 0.5 + \frac{1}{2} \cdot 2 = \frac{3}{2}, \tag{20}$$

and for quadratic interpolation, one obtains

$$p_2(0.5) = \frac{(0.5 - 1)(0.5 - 2)}{(-1) \cdot (-2)} \cdot 1 + \frac{(0.5 - 0)(0.5 - 2)}{1 \cdot (-1)} \cdot 2 + \frac{(0.5 - 0)(0.5 - 1)}{2 \cdot 1} \cdot 4 = \frac{11}{8} = 1.375. \tag{21}$$

(d) The method of Newton-Raphson is based on linearization around the iterate p_n. This is given by

$$L(x) = f(p_n) + (x - p_n) f'(p_n). \tag{22}$$

Next, we determine p_{n+1} such that $L(p_{n+1}) = 0$, that is

$$f(p_n) + (p_{n+1} - p_n) f'(p_n) = 0 \Leftrightarrow p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \qquad f'(p_n) \neq 0. \tag{23}$$

This result can also be proved graphically, see book, chapter 4.

(e) We have $f(x) = x^2 - 2x - 2$, so $f'(x) = 2x - 2$ and hence

$$p_{n+1} = p_n - \frac{p_n^2 - 2p_n - 2}{2p_n - 2}.$$

With the initial value $p_0 = 2$, this gives

$$p_1 = 2 - \frac{4 - 4 - 2}{4 - 2} = 3.$$

(f) We consider a Taylor polynomial around p_n, to express p

$$0 = f(p) = f(p_n) + (p - p_n) f'(p_n) + \frac{(p - p_n)^2}{2} f''(\xi_n), \tag{24}$$

for some ξ_n between p and p_n. Note that this form gives the exact representation. Subsequently, we consider the Newton-Raphson approximation

$$0 = L(p_{n+1}) = f(p_n) + (p_{n+1} - p_n)f'(p_n). \tag{25}$$

Subtraction of these two above equations gives

$$p_{n+1} - p = \frac{(p_n - p)^2}{2}\frac{f''(\xi_n)}{f'(p_n)}, \text{ provided that } f'(p_n) \neq 0, \tag{26}$$

and hence

$$|p_{n+1} - p| = \frac{(p_n - p)^2}{2}|\frac{f''(\xi_n)}{f'(p_n)}|, \text{ provided that } f'(p_n) \neq 0, \tag{27}$$

Using $p_n \to p$, $\xi_n \to p$ as $n \to \infty$ and continuity of $f(x)$ up to at least the second derivative, we arrive at $K = |\frac{f''(p)}{f'(p)}|$. □