DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Monday January 28 2013, 18:30-21:30

1. [a] The local truncation error is given by

$$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h}, \tag{1}$$

where z_{n+1} is computed by one step of the method starting from y_n, and we determine y_{n+1} by the use of a Taylor Series around t_n:

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2} y''(t_n) + O(h^3). \tag{2}$$

We realize that

$$\begin{aligned} y'(t_n) &= f(t_n, y_n) \\ y''(t_n) &= \frac{df(t_n, y_n)}{dt} = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} y'(t_n) = \\ &= \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} f(t_n, y_n). \end{aligned} \tag{3}$$

Hence, this gives

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2} \left(\frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} f(t_n, y_n) \right) + O(h^3). \tag{4}$$

For z_{n+1}, after substitution of the predictor–step for z^*_{n+1} into the corrector–step, and using the Taylor Series around (t_n, y_n)

$$\begin{aligned} z_{n+1} &= y_n + \tfrac{h}{2} \left(f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n)) \right) = \\ & y_n + \frac{h}{2} \left(f(t_n, y_n) + f(t_n, y_n) + h\Big(\frac{\partial f(t_n, y_n)}{\partial t} + f(t_n, y_n) \frac{\partial f(t_n, y_n)}{\partial y}\Big) + O(h^2) \right). \end{aligned} \tag{5}$$

Then, it follows that

$$y_{n+1} - z_{n+1} = O(h^3), \text{ and hence } \tau_{n+1}(h) = \frac{O(h^3)}{h} = O(h^2). \tag{6}$$

[b] Consider the test–equation $y' = \lambda y$, then it follows that

$$\begin{aligned} w_{n+1}^* &= w_n + h\lambda w_n = (1+h\lambda)w_n, \\ w_{n+1} &= w_n + \frac{h}{2}(\lambda w_n + \lambda w_{n+1}^*) = \\ &= w_n + \frac{h}{2}(\lambda w_n + \lambda(w_n + h\lambda w_n)) = (1 + h\lambda + \frac{(h\lambda)^2}{2})w_n. \end{aligned} \tag{7}$$

Hence the amplification factor is given by

$$Q(h\lambda) = 1 + h\lambda + \frac{(h\lambda)^2}{2}. \tag{8}$$

[c] Let $x_1 = y$ and $x_2 = y'$, then it follows that $y'' = x_2'$, and hence we get

$$\begin{aligned} & x_2' + 12x_2 + 72x_1 = \sin(t), \\ & x_2 = x_1'. \end{aligned} \tag{9}$$

This expression is written as

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -72x_1 - 12x_2 + \sin(t). \end{aligned} \tag{10}$$

Finally, we get the following matrix–form:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -72 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(t) \end{pmatrix}. \tag{11}$$

Here, we have $A = \begin{pmatrix} 0 & 1 \\ -72 & -12 \end{pmatrix}$ and $f = \begin{pmatrix} 0 \\ \sin(t) \end{pmatrix}$. The initial conditions are given by $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

[d] The Modified Euler Method, applied to the system $\underline{x}' = A\underline{x} + \underline{f}$, gives

$$\begin{aligned} \underline{w}_1^* &= \underline{w}_0 + h\left(A\underline{w}_0 + \underline{f}_0\right), \\ \underline{w}_1 &= \underline{w}_0 + \tfrac{h}{2}\left(A\underline{w}_0 + \underline{f}_0 + A\underline{w}_1^* + \underline{f}_1\right). \end{aligned} \tag{12}$$

With the initial condition and $h = 0.1$, this gives

$$\underline{w}_1^* = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{10}\left(\begin{pmatrix} 0 & 1 \\ -72 & -12 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1.2 \\ -7.6 \end{pmatrix}. \tag{13}$$

Then, the correction–step is given by

$$\underline{w}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \tfrac{1}{20}\left(\begin{pmatrix} 0 & 1 \\ -72 & -12 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -72 & -12 \end{pmatrix}\begin{pmatrix} 1.2 \\ -7.6 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(\frac{1}{10}) \end{pmatrix}\right) =$$

$$= \begin{pmatrix} 0.72 \\ -2.55501 \end{pmatrix} \tag{14}$$

[e] To this extent, we determine the eigenvalues of the matrix A. Subsequently, these eigenvalues are substituted into the amplification factor. The eigenvalues of A are given by $-6 \pm 6i$. Using $h = 0.25$, it follows that

$$Q(h\lambda) = 1+h\lambda+\frac{1}{2}h^2\lambda^2 = 1+\frac{1}{4}(-6+6i)+\frac{1}{32}(-6+6i)^2 = 1-\frac{3}{2}+\frac{3}{2}i-\frac{72}{32}i = -\frac{1}{2}-\frac{3}{4}i. \quad (15)$$

Herewith, it follows that $|Q(h\lambda)|^2 = \frac{1}{4} + \frac{9}{16} = \frac{13}{16} < 1$. Hence for $h = 0.25$, it follows that the method applied to the given system is stable. Note that this conclusion holds for both the eigenvalues of A since they are complex conjugates.

2. (a) The first order backward difference formula for the first derivative is given by

$$f'(t) \approx \frac{f(t) - f(t-h)}{h}.$$

Using $t = 2$, and $h = 1$ the approximation of the velocity is

$$\frac{f(2) - f(1)}{1} = 250 - 215 = 35 \text{ (m/s)}.$$

(b) Taylor polynomials are:

$$\begin{aligned} f(0) &= f(2h) - 2hf'(2h) + 2h^2 f''(2h) - \frac{(2h)^3}{6} f'''(\xi_0)\,, \\ f(h) &= f(2h) - hf'(2h) + \frac{h^2}{2} f''(2h) - \frac{h^3}{6} f'''(\xi_1)\,, \\ f(2h) &= f(2h). \end{aligned}$$

We know that $Q(h) = \frac{\alpha_0}{h} f(0) + \frac{\alpha_1}{h} f(h) + \frac{\alpha_2}{h} f(2h)$, which should be equal to $f'(2h) + O(h^2)$. This leads to the following conditions:

$$\begin{aligned} \tfrac{\alpha_0}{h} + \tfrac{\alpha_1}{h} + \tfrac{\alpha_2}{h} &= 0\,, \\ -2\alpha_0 - \alpha_1 &= 1\,, \\ 2\alpha_0 h + \tfrac{1}{2}\alpha_1 h &= 0\,. \end{aligned}$$

(c) The truncation error follows from the Taylor polynomials:

$$f'(2h) - Q(h) = f'(2h) - \frac{f(0) - 4f(h) + 3f(2h)}{2h} = \frac{\frac{8h^3}{6} f'''(\xi_0) - 4(\frac{h^3}{6} f'''(\xi_1))}{2h} = \frac{1}{3}h^2 f'''(\xi).$$

Using the new formula with $h = 1$ we obtain the estimate:

$$\frac{f(0) - 4f(1) + 3f(2)}{2} = \frac{200 - 4 \times 215 + 3 \times 250}{2} = 45 \text{ (m/s)}.$$

Note that the estimated velocity of the vehicle is larger than the maximum speed of 40 (m/s).

(d) To estimate the measuring error we note that

$$|Q(h) - \hat{Q}(h)| = |\frac{(f(0) - \hat{f}(0)) - 4(f(h) - \hat{f}(h)) + 3(f(2h) - \hat{f}(2h))}{2h}|$$

$$\leq \frac{|f(0) - \hat{f}(0)| + 4|f(h) - \hat{f}(h)| + 3|f(2h) - \hat{f}(2h)|}{2h} \leq \frac{4\epsilon}{h},$$

so $C_1 = 4$.

(e) We integrate $f(x)$, in which we approximate $f(x)$ by $p_1(x)$, then it follows:

$$\int_{x_0}^{x_1} f(x)dx \approx \int_{x_0}^{x_1} p_1(x)dx = \int_{x_0}^{x_1} \left\{ f(x_0)\frac{x - x_1}{x_0 - x_1} + f(x_1)\frac{x - x_0}{x_1 - x_0} \right\} dx =$$

$$= \left[\frac{1}{2}\frac{(x - x_0)^2}{x_1 - x_0} f(x_1) \right]_{x_0}^{x_1} + \left[\frac{1}{2}\frac{(x - x_1)^2}{x_0 - x_1} f(x_0) \right]_{x_0}^{x_1} = \frac{1}{2}(x_1 - x_0)(f(x_0) + f(x_1)). \tag{16}$$

This is the Trapezoidal Rule.

(f) The magnitude of the error of the numerical integration over interval $[x_0, x_1]$ is given by

$$|\int_{x_0}^{x_1} f(x)dx - \int_{x_0}^{x_1} p_1(x)dx| = |\int_{x_0}^{x_1} (f(x) - p_1(x))\, dx| =$$

$$|\int_{x_0}^{x_1} \frac{1}{2}(x - x_0)(x - x_1)f''(\chi(x))dx| \leq \frac{1}{2} \max_{x \in [x_0, x_1]} |f''(x)| \int_{x_0}^{x_1} (x - x_0)(x_1 - x)dx =$$

$$\frac{1}{12}(x_1 - x_0)^3 \max_{x \in [x_0, x_1]} |f''(x)|. \tag{17}$$