DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Thursday July 5 2012, 18:30-21:30

1. To integrate the initial value problem

$$y' = f(t, y(t)), \qquad y(t^0) = y^0, \tag{1}$$

we consider the Trapezoidal Rule

$$w^{n+1} = w^n + \frac{h}{2}(f(t^n, w^n) + f(t^{n+1}, w^{n+1})), \tag{2}$$

and the Modified Euler Method

$$\begin{cases} \hat{w}^{n+1} = w^n + hf(t^n, w^n), & \text{predictor–step,} \\ w^{n+1} = w^n + \frac{h}{2}(f(t^n, w^n) + f(t^{n+1}, \hat{w}^{n+1})), & \text{corrector–step.} \end{cases} \tag{3}$$

Here w^n denotes the numerical approximation at time $t^n = t^0 + nh$, and h represents the time–step.

[a] Use the test–equation to prove that the amplification factors of the two methods are given by

$$\begin{aligned} Q_T(h\lambda) &= \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}, && \text{Trapezoidal Rule,} \\ Q_{ME}(h\lambda) &= 1 + h\lambda + \tfrac{(h\lambda)^2}{2}, && \text{Modified Euler Method.} \end{aligned} \tag{4}$$

(2pt.)

[b] Show that the local truncation error of both methods is of order $O(h^2)$.

Hint: It is allowed to use the test equation for both the Trapezoidal Rule and Modified Euler Method. Further, note that $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + O(x^4)$ *and if* $|x| < 1$ *then* $\frac{1}{1-x} = 1 + x + x^2 + x^3 + O(x^4)$. (3pt.)

[0]**please turn over, For the answers of this test we refer to:**
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html

We apply both methods to the initial value problem

$$y'' + y = t(1-t), \quad y(0) = 0, \quad y'(0) = 1. \tag{5}$$

[c] Show that, by setting $y_1(t) = y(t)$ and $y_2(t) = y'(t)$, this initial value problem can be rewritten as the following system of first–order equations

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ t(1-t) \end{pmatrix}, \tag{6}$$

with initial condition $y_1(0) = 0$ and $y_2(0) = 1$. (1pt.)

[d] Use $h = \frac{1}{2}$ to compute w_1 (one time–step) using both the Trapezoidal Rule and the Modified Euler Method. (2pt.)

[e] Which of the two methods do you prefer to apply to the present initial value problem (in assignment [c–d])? Motivate your choice in terms of accuracy, stability and workload. (2pt.)

2. (a) The following iteration process is given $x_{n+1} = g(x_n)$, with

$$g(x_n) = x_n + h(x_n)(x_n^3 - 3),$$

where h is a continuous function with $h(x) \neq 0$ for each $x \neq 0$. If this process converges, to which (real valued) limit p does it converge? (1pt.)

(b) Consider three possible choices for $h(x)$:

i. $h_1(x) = -\frac{1}{x^4}$

ii. $h_2(x) = -\frac{1}{x^2}$

iii. $h_3(x) = -\frac{1}{3x^2}$

For which choice does the process not converge? For which choice is the convergence the fastest? Motivate your answer. (2pt.)

(c) p is the root of a given function f. $\hat{f}$ is the function perturbed by measurement errors. It is given that $|\hat{f}(x) - f(x)| \leq \epsilon_{max}$ for all x. Show that the root $\hat{p}$ from $\hat{f}$ satisfies the following inequality $|\hat{p} - p| \leq \frac{\epsilon_{max}}{|f'(p)|}$. (1pt.)

(d) Subsequently, we use the Newton-Raphson method, which is given by

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}.$$

We take $f(x) = x^4 - 3x$. Do one step with the Newton-Raphson method with initial guess $z_0 = 1$. (2 pt.)

(e) Derive the Newton-Raphson method. (2 pt.)

(f) Let z be the solution of $f(z) = 0$. Demonstrate that

$$|z - z_{k+1}| = K|z - z_k|^2, \text{ for } k \to \infty \tag{7}$$

and determine the value of K (for $k \to \infty$). (2 pt.)