DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Thursday August 30 2012, 18:30-21:30

1. We consider the following predictor-corrector method for the integration of the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$:

$$\begin{aligned} w_{n+1}^* &= w_n + hf(t_n, w_n), \\ w_{n+1} &= w_n + h\left((1-\mu)f(t_n, w_n) + \mu f(t_{n+1}, w_{n+1}^*)\right), \end{aligned} \tag{1}$$

where h, μ and w_n respectively denote the time step, a real number ($0 \le \mu \le 1$), and the numerical solution at time t_n.

(a) Show that the local truncation error of the abovementioned method is of order $O(h)$, if $0 \le \mu \le 1$ and of order $O(h^2)$, if $\mu = \frac{1}{2}$ (*Note that this has to be demonstrated for the general differential equation* $y' = f(t, y)$). (3 pt)

(b) Demonstrate that the amplification factor of the abovementioned method, is given by

$$Q(h\lambda) = 1 + h\lambda + \mu(h\lambda)^2.$$

(2 pt)

(c) We consider the following system of non linear differential equations:

$$\begin{aligned} x_1' &= -\sin x_1 + 2x_2 + t, \ \ x_1(0) = 0, \\ x_2' &= x_1 - x_2^2, \ \ x_2(0) = 1. \end{aligned} \tag{2}$$

Do one step with the method given in (1) with $h = \frac{1}{2}$ and $\mu = \frac{1}{2}$. (1 pt)

(d) Show that the Jacobian of the right-hand side of (2) at $t = 0$ is given by:

$$\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}.$$

(1 pt)

(e) Choose $\mu = 0$. For which values of h is the method applied to (2) stable at $t = 0$? Answer the same question for $\mu = \frac{1}{2}$. (3 pt)

[0]**please turn over, For the answers of this test we refer to: http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html**

2. We approximate the integral $\int_a^b f(x)dx$ using gridnodes $x_j = a + (j-1)h$, where $x_{n+1} = b$. For an interval between two adjacent gridnodes, (x_j, x_{j+1}), the Rectangle Rule gives the approximation $hf(x_j)$ and a repetitive application gives $\int_a^b f(x)dx \approx T_0 = h\sum_{j=1}^n f(x_j)$.

   a Show that the local truncation error, $|E_0^I| := |\int_{x_j}^{x_{j+1}} f(x)dx - hf(x_j)|$, and global error, $|E_0| := |\int_a^b f(x)dx - T_0|$ are, respectively, given by

$$|E_0^I| \leq \frac{h^2}{2} \max_{x\in[x_j,x_{j+1}]} |f'(x)|, \text{ and } |E_0| \leq \frac{(b-a)h}{2} \max_{x\in[a,b]} |f'(x)|. \tag{3}$$

   Hint: You can use Taylor's Theorem, and $\max_{x\in[a,b]} |f(x)|$ *denotes the maximum of* $|f(x)|$ *over the interval* $[a,b]$. (2 pt.)

   (a) Next we also incorporate the first–order derivative of f in the gridnodes $\{x_j\}$. Use Taylor's Theorem to derive that the integral can be approximated by T_1 using the gridnodes with global error E_1, where

$$\int_a^b f(x)dx \approx T_1 = h\sum_{j=1}^n \left[f(x_j) + \frac{h}{2}f'(x_j)\right], \quad |E_1| \leq \frac{(b-a)h^2}{6} \max_{x\in[a,b]} |f''(x)|. \tag{4}$$

   (2pt.)

   (b) Use the method from equation (4) with $h = \frac{1}{2}$ to approximate $\int_0^1 x^2 dx$ and compare the error with the estimate for $|E_1|$. (2pt.)

   d Next we use the derivatives of f up to the order 2. Further, T_2 and E_2 are, respectively, the approximation of $\int_a^b f(x)dx$ using these derivatives and the corresponding global error. Show that

$$T_2 = T_1 + \frac{h^3}{3!}\sum_{j=1}^n f''(x_j), \text{ and } |E_2| \leq \frac{(b-a)h^3}{4!} \max_{x\in[a,b]} |f'''(x)|. \tag{5}$$

   (2pt.)

   (c) Suppose that all values of f and their derivatives contain a error of measurement or rounding, i.e. $|\tilde{f}^{(k)}(x_j) - f^{(k)}(x_j)| \leq \varepsilon$, for all j and k–th derivatives ($k = 0$ gives f itself). Let T_2 and $\tilde{T}_2$, respectively, be computed using the exact (f) and available values ($\tilde{f}$) of f and its derivatives, show that the influence of the this error can be estimated by:

$$|\tilde{T}_2 - T_2| \leq (b-a)\varepsilon\left(1 + \frac{h}{2} + \frac{h^2}{3!}\right). \tag{6}$$

   (2pt.)