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**ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)**
**Thursday August 30 2012, 18:30-21:30**

1. (a) The local truncation error is given by

$$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h} \tag{1}$$

where $z_{n+1}$ is the result of applying the method once with starting solution $y_n$. Here we obtain $y_{n+1}$ by a Taylor expansion around $t_n$:

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2}y''(t_n) + O(h^3). \tag{2}$$

For $z_{n+1}$, we obtain, after substitution of the predictor step for $z^*_{n+1}$ into the corrector step

$$z_{n+1} = y_n + h\left((1-\mu)f(t_n, y_n) + \mu f(t_n + h, y_n + hf(t_n, y_n))\right) \tag{3}$$

After a Taylor expansion of $f(t_n+h, y_n+hf(t_n, y_n))$ around $(t_n, y_n)$ one obtains:

$$z_{n+1} = y_n+h\left((1-\mu)f(t_n, y_n) + \mu(f(t_n, y_n) + h(\frac{\partial f(t_n, y_n)}{\partial t} + f(t_n, y_n)\frac{\partial f(t_n, y_n)}{\partial y})) + O(h^2)\right). \tag{4}$$

From the differential equation we know that:

$$y'(t_n) = f(t_n, y_n) \tag{5}$$

From the Chain Rule of Differentiation, we derive

$$y''(t_n) = \frac{df(t_n, y_n)}{dt} = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}y'(t_n) \tag{6}$$

after substitution of the differential equation one obtains:

$$y''(t_n) = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}f(t_n, y_n) \tag{7}$$

This implies that $z_{n+1} = y_n + hy'(t_n) + \mu h^2 y''(t_n)$. Subsequently, it follows that

$$y_{n+1} - z_{n+1} = O(h^2), \text{ and, hence } \tau_{n+1}(h) = \frac{O(h^2)}{h} = O(h) \text{ for } 0 \leq \mu \leq 1, \tag{8}$$

$$y_{n+1} - z_{n+1} = O(h^3), \text{ and, hence } \tau_{n+1}(h) = \frac{O(h^3)}{h} = O(h^2) \text{ for } \mu = \frac{1}{2}. \tag{9}$$

(b) Consider the test equation $y' = \lambda y$, then, herewith, one obtains

$$\begin{aligned} w_{n+1}^* &= w_n + h\lambda w_n = (1+h\lambda)w_n, \\ w_{n+1} &= w_n + h((1-\mu)\lambda w_n + \mu\lambda w_{n+1}^*) = \\ &= w_n + h((1-\mu)\lambda w_n + \mu\lambda(w_n + h\lambda w_n)) = (1+h\lambda+\mu(h\lambda)^2)w_n. \end{aligned} \tag{10}$$

Hence the amplification factor is given by

$$Q(h\lambda) = 1 + h\lambda + \mu(h\lambda)^2. \tag{11}$$

(c) Doing one step with the given method with $h = \frac{1}{2}$ and $\mu = \frac{1}{2}$ leads to the following steps:
Predictor:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\sin(0)+2+0 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$$

Corrector:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2}\left(\frac{1}{2}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\sin(1)+2\cdot\frac{1}{2}+\frac{1}{2} \\ 1-(\frac{1}{2})^2 \end{pmatrix}\right)$$

which can be written as:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0+\frac{1}{2}-\frac{1}{4}\sin(1)+\frac{3}{8} \\ 1-\frac{1}{4}+\frac{1}{4}-\frac{1}{16} \end{pmatrix} = \begin{pmatrix} \frac{7}{8}-\frac{1}{4}\sin(1) \\ \frac{15}{16} \end{pmatrix} = \begin{pmatrix} 0.6646 \\ 0.9375 \end{pmatrix}$$

(d) In order to compute the Jacobian, we note that the right-hand side of the non linear system can be noted by:

$$f_1(x_1, x_2) = -\sin x_1 + 2x_2 + t$$

$$f_2(x_1, x_2) = x_1 - x_2^2$$

From the definition of the Jacobian it follows that:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -\cos x_1 & 2 \\ 1 & -2x_2 \end{pmatrix}.$$

Substitution of $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ shows that

$$J = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}.$$

(e) For the stability it is sufficient to check that $|Q(h\lambda_i)| \leq 1$ for all the eigenvalues of the Jacobian matrix. It is easy to see that the eigenvalues of the Jacobian matrix are $\lambda_1 = -3$ and $\lambda_2 = 0$.

For the choice $\mu = 0$ we note that the method is equal to the Euler Forward method. For real eigenvalues the Euler Forward method is stable if $h \leq \frac{-2}{\lambda}$. Since $\lambda_1 = -3$ and $\lambda_2 = 0$ we know that the method is stable if $h \leq \frac{-2}{-3} = \frac{2}{3}$ (another option is to derive the values of $h$ such that $|Q(h\lambda_i)| \leq 1$ by using the description of $Q(h\lambda)$)

For the choice $\mu = \frac{1}{2}$ we use the expression

$$Q(h\lambda) = 1 + h\lambda + \frac{1}{2}(h\lambda)^2$$

For $\lambda_2 = 0$ it appears that $Q(h\lambda_2) = 1$ so the inequality is satisfied for all $h$. For $\lambda_1 = -3$ we have to check the following inequalities:

$$-1 \leq 1 - 3h + \frac{9}{2}h^2 \leq 1$$

For the left-hand inequality we arrive at

$$0 \leq \frac{9}{2}h^2 - 3h + 2$$

It appears that the discriminant $9 - 4 \cdot \frac{9}{2} \cdot 2$ is negative, so there are no real roots which implies that the inequality is satisfied for all $h$.

For the right-hand inequality we get

$$-3h + \frac{9}{2}h^2 \leq 0$$

$$\frac{9}{2}h^2 \leq 3h$$

so

$$h \leq \frac{2}{3}$$

(another option is to see that for $\mu = \frac{1}{2}$ the method is equal to the modified Euler method, and remember that this method is stable for real eigenvalues if $h \leq \frac{-2}{\lambda}$)

2. (a) Taylor's Theorem (or here the Mean Value Theorem) gives for a zeroth order approximation around $x_j$:

$$f(x) = f(x_j) + (x - x_j)f'(\xi(x)), \tag{12}$$

for a $\xi(x) \in (x_j, x)$ if $x > x_j$. Then we consider the interval $[x_j, x_{j+1})$ and use Taylor's Theorem around $x_j$ in the integration to get

$$\int_{x_j}^{x_{j+1}} f(x)dx = \int_{x_j}^{x_{j+1}} f(x_j) + (x - x_j)f'(\xi(x))dx = hf(x_j) + \int_{x_j}^{x_{j+1}} (x - x_j)f'(\xi(x))dx. \tag{13}$$

Hence we get

$$|\int_{x_j}^{x_{j+1}} f(x)dx - \int_{x_j}^{x_{j+1}} f(x_j)| = |\int_{x_j}^{x_{j+1}} (x-x_j)f'(\xi(x))dx|. \tag{14}$$

Taking the maximum value of $f'$ over the interval $[x_j, x_{j+1}]$, yields

$$|\int_{x_j}^{x_{j+1}} (x-x_j)f'(\xi(x))dx| \leq \max_{x\in[x_j,x_{j+1}]} |f'(x)| \int_{x_j}^{x_{j+1}} (x-x_j)dx = \frac{h^2}{2} \max_{x\in[x_j,x_{j+1}]} |f'(x)|. \tag{15}$$

By combining relations (14) and (15), we proved that

$$|\int_{x_j}^{x_{j+1}} f(x)dx - \int_{x_j}^{x_{j+1}} f(x_j)| \leq \frac{h^2}{2} \max_{x\in[x_j,x_{j+1}]} |f'(x)|. \tag{16}$$

Next, we deal with the entire interval $[a, b]$, then

$$|\int_a^b f(x)dx - h\sum_{j=1}^{n} f(x_j)| = |\sum_{j=1}^{n} \left( \int_{x_j}^{x_{j+1}} f(x)dx - hf(x_j) \right)|. \tag{17}$$

We use the Triangle Inequality to get

$$|\sum_{j=1}^{n} \left( \int_{x_j}^{x_{j+1}} f(x)dx - hf(x_j) \right)| \leq \sum_{j=1}^{n} |\int_{x_j}^{x_{j+1}} f(x)dx - hf(x_j)|. \tag{18}$$

From relation (16), it follows that

$$\sum_{j=1}^{n} |\int_{x_j}^{x_{j+1}} f(x)dx - hf(x_j)|. \leq \frac{h^2}{2} \sum_{j=1}^{n} \max_{x\in[x_j,x_{j+1}]} |f'(x)|. \tag{19}$$

Since $\max_{x\in([a,b]} |f'(x)| \geq \max_{x\in([x_j,x_{j+1}]} |f'(x)|,\ \forall j \in \{1, \ldots, n\}$, we get

$$\frac{h^2}{2} \sum_{j=1}^{n} \max_{x\in[x_j,x_{j+1}]} |f'(x)| \leq \frac{h^2}{2} \cdot n \cdot \max_{x\in[a,b]} |f'(x)|. \tag{20}$$

Since $x_{n+1} = a + nh = b$, we have $nh = b - a$ and hence the above inequality gives

$$\frac{h^2}{2} \sum_{j=1}^{n} \max_{x\in[x_j,x_{j+1}]} |f'(x)| \leq \frac{h^2}{2} \cdot n \cdot \max_{x\in[a,b]} |f'(x)| = \frac{h}{2}(b-a) \max_{x\in[a,b]} |f'(x)|. \tag{21}$$

Hence the global error can be estimated from above by

$$|\int_a^b f(x)dx - h\sum_{j=1}^{n} f(x_j)| \leq \frac{h}{2}(b-a) \max_{x\in[a,b]} |f'(x)|. \tag{22}$$

(b) Incorporating the first–order derivative in Taylor's Theorem (linearization) gives

$$f(x) = f(x_j) + (x - x_j)f'(x_j) + \frac{(x - x_j)^2}{2} f''(\xi(x)), \tag{23}$$

for a $\xi(x) \in (x_j, x)$ if $x > x_j$. We start integrating over the interval $[x_j, x_{j+1}]$ to get

$$\int_{x_j}^{x_{j+1}} f(x)dx = \int_{x_j}^{x_{j+1}} f(x_j) + (x - x_j)f'(x_j) + \frac{(x - x_j)^2}{2} f''(\xi(x))dx = \\ hf(x_j) + \frac{h^2}{2} f'(x_j) + \int_{x_j}^{x_{j+1}} \frac{(x - x_j)^2}{2} f''(\xi(x))dx. \tag{24}$$

Hence, we obtain

$$|\int_{x_j}^{x_{j+1}} f(x)dx - (hf(x_j) + \frac{h^2}{2} f'(x_j))| = |\int_{x_j}^{x_{j+1}} \frac{(x - x_j)^2}{2} f''(\xi(x))dx| \leq \\ \max_{x \in [x_j, x_{j+1}]} |f''(x)| \int_{x_j}^{x_{j+1}} \frac{(x - x_j)^2}{2} = \frac{h^3}{6} \max_{x \in [x_j, x_{j+1}]} |f''(x)|. \tag{25}$$

Analogously to the previous assignment, we get

$$|E_1| = |\int_a^b f(x)dx - \sum_{j=1}^{n} \left(hf(x_j) + \frac{h^2}{2} f'(x_j)\right)| = |\sum_{j=1}^{n} \left(\int_{x_j}^{x_{j+1}} f(x)dx - \left(hf(x_j) + \frac{h^2}{2} f'(x_j)\right)\right)| \leq \\ \sum_{j=1}^{n} |\left(\int_{x_j}^{x_{j+1}} f(x)dx - \left(hf(x_j) + \frac{h^2}{2} f'(x_j)\right)\right)| \leq \frac{h^3}{6} \sum_{j=1}^{n} \max_{x \in [x_j, x_{j+1}]} |f''(x)| \leq \\ \frac{h^3}{6} \cdot n \cdot \max_{x \in [a,b]} |f''(x)| = \frac{h^2}{6}(b - a) \max_{x \in [a,b]} |f''(x)|. \tag{26}$$

Hence $\int_a^b f(x)dx \approx \sum_{j=1}^{n} h(f(x_j) + \frac{h}{2} f'(x_j)) = T_1$ where the global error is estimated from above by the above expression.

(c) Upon considering the interval $(0, 1)$ with $h = \frac{1}{2}$, we use $x_1 = 0$ and $x_2 = \frac{1}{2}$ $(n = 2)$. Then, we get

$$\int_0^1 x^2 dx \approx h(f(x_1) + f(x_2) + \frac{h}{2}(f'(x_1) + f'(x_2)) = \frac{1}{2}(0 + (\frac{1}{2})^2 + \frac{1}{4}(0 + 2 \cdot \frac{1}{2})) = \frac{1}{4}. \tag{27}$$

The exact answer is given by $\frac{1}{3}$, hence the error is $\frac{1}{12}$. To check our result, we use the upper bound of the error given in relation (26):

$$\frac{h^2}{6}(b - a) \max_{x \in [a,b]} |f''(x)| = \frac{1}{6} \cdot (\frac{1}{2})^2 \cdot 1 \cdot 2 = \frac{1}{12}. \tag{28}$$

Note that here it was used that the second–order derivative of $x^2$ is given by 2. Hence our the error that we found using the exact solution does not exceed the upper bound from relation (26), and hence our result makes sense.

(d) $T_1$ is the approximation of the integral obtained by the use the first order derivatives, hence $T_2$ is the analogon with the first and second order derivatives, hence

$$\begin{aligned} &T_2 = \\ &\sum_{j=1}^{n}\left(\int_{x_j}^{x_{j+1}} f(x_j) + (x-x_j)f'(x_j) + \frac{(x-x_j)^2}{2}f''(x_j)dx\right) = \\ &\sum_{j=1}^{n}\left(\int_{x_j}^{x_{j+1}} f(x_j) + (x-x_j)f'(x_j)dx\right) + \sum_{j=1}^{n}\int_{x_j}^{x_{j+1}} \frac{(x-x_j)^2}{2}f''(x_j)dx \\ &= T_1 + \sum_{j=1}^{n}\int_{x_j}^{x_{j+1}} \frac{(x-x_j)^2}{2}f''(x_j)dx = T_1 + \frac{h^3}{3!}\sum_{j=1}^{n} f''(x_j). \end{aligned} \tag{29}$$

The last step follows from evaluation of the integral. Hence we demonstrated that

$$T_2 = T_1 + \frac{h^3}{3!}\sum_{j=1}^{n} f''(x_j). \tag{30}$$

Further, the local error is found by using Taylor's Theorem over the interval $[x_j, x_{j+1}]$ to get

$$\begin{aligned} &\left|\int_{x_j}^{x_{j+1}} f(x)dx - \left(\int_{x_j}^{x_{j+1}} f(x_j) + \ldots + \frac{(x-x_j)^2}{2!}f''(x_j)dx\right)\right| = \\ &\left|\int_{x_j}^{x_{j+1}} \frac{(x-x_j)^3}{3!}f'''(\xi(x))dx\right| \leq \max_{x\in[x_j,x_{j+1}]}|f'''(x)|\int_{x_j}^{x_{j+1}} \frac{(x-x_j)^3}{3!}dx = \\ &\frac{h^4}{4!}\max_{x\in[x_j,x_{j+1}]}|f'''(x)|. \end{aligned} \tag{31}$$

Here, the last step follows from evaluation of the integral. A summation procedure over all intervals, similar to assignment 2.a., gives the global error bound:

$$\begin{aligned} &|E_2| = \left|\int_a^b f(x)dx - T_2\right| \leq \frac{h^4}{4!}\sum_{j=1}^{n}\max_{x\in[x_j,x_{j+1}]}|f'''(x)| \leq \\ &\frac{h^4}{4!}\cdot n \cdot \max_{x\in[a,b]}|f'''(x)| = \frac{h^3(b-a)}{4!}\max_{x\in[a,b]}|f'''(x)|. \end{aligned} \tag{32}$$

(e) Let $T_2$ and $\tilde{T}_2$, respectively, be the approximation of $\int_a^b f(x)dx$ using the exact and available values of $f$ and its derivatives. Then, we have

$$T_2 = \sum_{j=1}^{n} \int_{x_j}^{x_{j+1}} f(x_j) + \ldots + \frac{(x-x_j)^2}{2!} f''(x_j) dx =$$

$$\sum_{j=1}^{n} \left( hf(x_j) + \frac{h^2}{2} f'(x_j) + \frac{h^3}{3!} f''(x_j) \right) = \tag{33}$$

$$h \sum_{j=1}^{n} f(x_j) + \frac{h^2}{2} \sum_{j=1}^{n} f'(x_j) + \frac{h^3}{3!} \sum_{j=1}^{n} f''(x_j).$$

For $\tilde{T}_2$, we similarly have

$$\tilde{T}_2 = h \sum_{j=1}^{n} \tilde{f}(x_j) + \frac{h^2}{2} \sum_{j=1}^{n} \tilde{f}'(x_j) + \frac{h^3}{3!} \sum_{j=1}^{n} \tilde{f}''(x_j). \tag{34}$$

Subtraction of the above two equations, taking the absolute value, and using the Triangle Inequality, gives

$$|T_2 - \tilde{T}_2| \leq$$

$$h \sum_{j=1}^{n} |f(x_j) - \tilde{f}(x_j)| + \frac{h^2}{2} \sum_{j=1}^{n} |f'(x_j) - \tilde{f}'(x_j)| + \frac{h^3}{3!} \sum_{j=1}^{n} |f''(x_j) - \tilde{f}''(x_j)|, \tag{35}$$

Using $|f^{(k)}(x_j) - \tilde{f}^{(k)}(x_j)| \leq \varepsilon$ for all $k$ and $j$, and $nh = b - a$, gives

$$|T_2 - \tilde{T}_2| \leq h \cdot n \cdot \varepsilon + \frac{h^2}{2} \cdot n \cdot \varepsilon + \frac{h^3}{3!} \cdot n \cdot \varepsilon =$$

$$(b-a)\varepsilon \left( 1 + \frac{h}{2} + \frac{h^2}{3!} \right) = (b-a)\varepsilon \sum_{k=1}^{3} \frac{h^{k-1}}{k!}. \tag{36}$$

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