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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
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1.

a The local truncation error is defined as

$$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h}, \tag{1}$$

where z_{n+1} is given by

$$z_{n+1} = y_n + h\left(a_1 f(t_n, y_n) + a_2 f(t_n + h, y_n + hf(t_n, y_n))\right). \tag{2}$$

A Taylor expansion of f around (t_n, y_n) yields

$$f(t_n+h, y_n+hf(t_n, y_n)) = f(t_n, y_n)+h\frac{\partial f}{\partial t}(t_n, y_n)+hf(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n)+O(h^2). \tag{3}$$

This is substituted into equation (2) to obtain

$$z_{n+1} = y_n+h\left(a_1 f(t_n, y_n) + a_2\left[f(t_n, y_n) + h\frac{\partial f}{\partial t}(t_n, y_n) + hf(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n)\right]\right)+O(h^3). \tag{4}$$

A Taylor series for $y(x)$ around t_n gives for y_{n+1}

$$y_{n+1} = y(t_n + h) = y_n + hy'(t_n) + \frac{h^2}{2}y''(t_n) + O(h^3). \tag{5}$$

From the differential equation we know that:

$$y'(t_n) = f(t_n, y_n) \tag{6}$$

From the Chain Rule of Differentiation, we derive

$$y''(t_n) = \frac{df(t_n, y_n)}{dt} = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}y'(t_n) \tag{7}$$

after substitution of the differential equation one obtains:

$$y''(t_n) = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}f(t_n, y_n) \tag{8}$$

Equations (5) and (4) are substituted into relation (1) to obtain

$$\tau_{n+1}(h) = f(t_n, y_n)(1 - (a_1 + a_2)) + h\left(\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right)\left(\frac{1}{2} - a_2\right) + O(h^2) \tag{9}$$

Hence

(a) $a_1 + a_2 = 1$ implies $\tau_{n+1}(h) = O(h)$;

(b) $a_1 + a_2 = 1$ and $a_2 = 1/2$, that is, $a_1 = a_2 = 1/2$, gives $\tau_{n+1}(h) = O(h^2)$.

b The test equation is given by

$$y' = \lambda y. \tag{10}$$

Application of the predictor step to the test equation gives

$$w^*_{n+1} = w_n + h\lambda w_n = (1 + h\lambda)w_n. \tag{11}$$

The corrector step yields

$$w_{n+1} = w_n + h\left(a_1\lambda w_n + a_2\lambda(1 + h\lambda)w_n\right) = (1 + (a_1 + a_2)h\lambda + a_2h^2\lambda^2)w_n. \tag{12}$$

Hence the amplification factor is given by

$$Q(h\lambda) = 1 + (a_1 + a_2)h\lambda + a_2h^2\lambda^2. \tag{13}$$

c Let $\lambda < 0$ (so λ is real), then, for stability, the amplification factor must satisfy

$$-1 \leq Q(h\lambda) \leq 1, \tag{14}$$

from the previous assignment, we have

$$-1 \leq 1 + (a_1 + a_2)h\lambda + a_2(h\lambda)^2 \leq 1 \Leftrightarrow -2 \leq (a_1 + a_2)h\lambda + a_2(h\lambda)^2 \leq 0. \tag{15}$$

First, we consider the left inequality:

$$a_2(h\lambda)^2 + (a_1 + a_2)h\lambda + 2 \geq 0 \tag{16}$$

For $h\lambda = 0$, the above inequality is satisfied, further the discriminant is given by $(a_1 + a_2)^2 - 8a_2 < 0$. Here the last inequality follows from the given hypothesis. Hence the left inequality in relation (15) is always satisfied. Next we consider the right hand inequality of relation (15)

$$a_2(h\lambda)^2 + (a_1 + a_2)h\lambda \leq 0. \tag{17}$$

This relation is rearranged into

$$a_2(h\lambda)^2 \leq -(a_1 + a_2)h\lambda, \tag{18}$$

hence

$$a_2|h\lambda|^2 \leq (a_1 + a_2)|h\lambda| \Leftrightarrow |h\lambda| \leq \frac{a_1 + a_2}{a_2}, \qquad a_2 \neq 0. \tag{19}$$

This results into the following condition for stability

$$h \leq \frac{a_1 + a_2}{a_2|\lambda|}, \qquad a_2 \neq 0. \tag{20}$$

d The Jacobian, J, is given by

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} \\ \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} \end{pmatrix}. \tag{21}$$

Since $f_1(y_1, y_2) = -y_1 y_2$ and $f_2(y_1, y_2) = y_1 y_2 - y_2$, we obtain

$$J = \begin{pmatrix} -y_2 & -y_1 \\ y_2 & y_1 - 1 \end{pmatrix}. \tag{22}$$

Substitution of the initial values $y_1(0) = 1$ and $y_2(0) = 2$, gives

$$J = \begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix}. \tag{23}$$

e The eigenvalues of the Jacobian at $y_1(0) = y_2(0) = 1$ are given by $\lambda_{1,2} = 1 \pm i$. For our case, we have

$$Q(h\lambda) = -1 + h\lambda + 1/2(h\lambda)^2. \tag{24}$$

Since our eigenvalues are not real valued, it is required for stability that

$$|Q(h\lambda)| \leq 1. \tag{25}$$

Since the eigenvalues are complex conjugates, we can proceed with one of the eigenvalues, say $\lambda = -1 + i$ with $\lambda^2 = -2i$ to obtain

$$Q(h\lambda) \quad = 1 + h(-1 + i) + 1/2h^2(-2i) \tag{26}$$

Substitution of $h = 1$ shows that $Q(h\lambda) = 0$. This implies that $|Q(h\lambda)| = 0 \leq 1$ so the method is stable.

2. a Given $v(x) = x(2 - x)$, then $v''(x) = -2$, and hence $-v'' + v = 2 + x(2 - x)$ follows by simple addition. Further, $v(0) = 0$ and $v'(x) = 2 - 2x$ and hence $v'(1) = 0$. Hence the differential equation, as well as the boundary conditions are satisfied.

   b Let $v_j = v(x_j)$, and let $x_n = 1$, hence $h = 1/n$, then

$$v_{j-1} = v(x_j - h) = v_j - hv'(x_j) + h^2/2v''(x_j) - h^3/3!v'''(x_j) + h^4/4!v''''(x_j) + O(h^5);$$

$$v_{j+1} = v(x_j + h) = v_j + hv'(x_j) + h^2/2v''(x_j) + h^3/3!v'''(x_j) + h^4/4!v''''(x_j) + O(h^5). \tag{27}$$

   From the above expression, it can be seen that

$$v''(x_j) = \frac{v_{j-1} - 2v_j + v_{j+1}}{h^2} + \frac{h^2}{12}v''''(x_j) + O(h^3), \tag{28}$$

and hence the error is $O(h^2)$. This gives the following discretization

$$\frac{-w_{j-1}+2w_j-w_{j+1}}{h^2}+w_j=2+x_j(2-x_j), \qquad \text{for } j=1\ldots n, \tag{29}$$

where $x_j = jh$ and $w_j \approx v_j$ as the numerical (finite difference) solution under neglecting the error. Further, we use a virtual gridnode near $x=1$, $x_{n+1}=1+h$, with

$$0=v'(1)=\frac{v_{n+1}-v_{n-1}}{2h}-\frac{h^2}{3}v'''(1)+O(h^3), \tag{30}$$

hence the error is $O(h^2)$. Neglecting the error, and substitution into the discretization equation $j=n$, gives

$$\frac{-2w_{n-1}+2w_n}{h^2}+w_n=3. \tag{31}$$

Division by 2 to make the discretization symmetric, gives

$$\frac{-w_{n-1}+w_n}{h^2}+\frac{1}{2}w_n=\frac{3}{2}. \tag{32}$$

The boundary condition at $x=0$, gives

$$\frac{2w_1-w_2}{h^2}+w_1=2+h(2-h).. \tag{33}$$

c For $j=1$, we get, using $h=1/3$,

$$18w_1-9w_2+w_1=2+1/3*5/3=23/9. \tag{34}$$

For $j=2$, we obtain

$$-9w_1+18w_2-9w_3+w_2=26/9. \tag{35}$$

For $j=3=n$, we use $w_4=w_2$, which gives

$$-9w_2+9w_3+1/2w_3=3/2. \tag{36}$$

Hence, the system of equations is

$$\begin{cases} 19w_1-9w_2=23/9, \\ -9w_1+19w_2-9w_3=26/9, \\ -9w_2+19/2w_3=3/2. \end{cases} \tag{37}$$

d The exact solution is given by $v(x)=x(2-x)$, and hence all derivatives of order three and larger are zero. Further, the error is determined by the derivatives of third order and larger. This implies that the error is zero.

e To this extent, we consider the determination of the zeros of the following system of equations

$$\begin{cases} F_1(v_1, v_2) = 18v_1 - 9v_2 + v_1^2 - \frac{20}{9}, \\ \\ F_2(v_1, v_2) = -9v_1 + 18v_2 + v_2^2 - \frac{20}{9}. \end{cases}$$

We consider (v_1^k, v_2^k) as the kth estimate of the successive approximations. Linearization around the estimate (v_1^k, v_2^k) gives the following Newton method:

$$\frac{\partial(F_1, F_2)}{\partial(v_1, v_2)}(v_1^k, v_2^k) \begin{pmatrix} v_1^{k+1} - v_1^k \\ \\ v_2^{k+1} - v_2^k \end{pmatrix} = -\underline{F}(v_1^k, v_2^k), \tag{38}$$

where $\underline{F}(v_1, v_2) = [F_1(v_1, v_2)\ F_2(v_1, v_2)]^T$, and

$$\frac{\partial(F_1, F_2)}{\partial(v_1, v_2)}(v_1^k, v_2^k) = \begin{pmatrix} \frac{\partial F_1}{\partial v_1}(v_1^k, v_2^k) & \frac{\partial F_1}{\partial v_2}(v_1^k, v_2^k) \\ \\ \frac{\partial F_2}{\partial v_1}(v_1^k, v_2^k) & \frac{\partial F_2}{\partial v_2}(v_1^k, v_2^k) \end{pmatrix} = \begin{pmatrix} 18 + 2v_1^k & -9 \\ \\ -9 & 18 + 2v_2^k \end{pmatrix}, \tag{39}$$

is the Jacobian matrix. Using $v_1^0 = v_2^0 = 0$, we get

$$\begin{pmatrix} 18 & -9 \\ \\ -9 & 18 \end{pmatrix} \begin{pmatrix} v_1^1 - v_1^0 \\ \\ v_2^1 - v_2^0 \end{pmatrix} = \begin{pmatrix} 20/9 \\ \\ 20/9 \end{pmatrix}. \tag{40}$$

The solution is given by $v_1^1 - v_1^0 = 20/81 = v_2^1 - v_2^0$, and hence $v_1^1 = v_2^1 = 20/81$.