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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
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1. a The local truncation error is defined by

$$\tau_h = \frac{y_{n+1} - z_{n+1}}{h}, \tag{1}$$

where

$$z_{n+1} = y_n + hf(t_n, y_n), \tag{2}$$

for the forward Euler method. A Taylor expansion for y_{n+1} around t_n is given by

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2}y''(\xi), \quad \exists\, \xi \in (t_n, t_{n+1}). \tag{3}$$

Since $y'(t_n) = f(t_n, y_n)$, we use equation (1), to get

$$\tau_h = \frac{h}{2}y''(\xi), \quad \exists\, \xi \in (t_n, t_{n+1}). \tag{4}$$

Hence, the truncation error is of first order.

b We define $y_1 := y$ and $y_2 := y'$, hence $y_1' = y_2$. Further, we use the differential equation to obtain

$$y'' + \varepsilon y' + y = y_1'' + \varepsilon y_1' + y_1 = y_2' + \varepsilon y_2 + y_1. \tag{5}$$

Hence, we obtain

$$y_2' = -y_1 - \varepsilon y_2 + sin(t). \tag{6}$$

Hence the system is given by

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -y_1 - \varepsilon y_2 + sin(t). \end{aligned} \tag{7}$$

The initial conditions are given by

$$\begin{aligned} 1 &= y(0) = y_1(0), \\ 0 &= y'(0) = y_1'(0) = y_2(0). \end{aligned} \tag{8}$$

c First, we use the test equation, $y' = \lambda y$, to analyze numerical stability. For forward Euler, we obtain

$$w_{n+1} = w_n + h\lambda w_n = Q(h\lambda)w_n, \tag{9}$$

hence the amplification factor becomes

$$Q(h\lambda) = 1 + h\lambda. \tag{10}$$

The numerical solution is stable if and only if $|Q(h\lambda)| \leq 1$. Next, we deal with the case $\varepsilon = 0$, to obtain the following system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{11}$$

This system gives the following eigenvalues $\lambda_{1,2} = \pm i$, where i is the imaginary unit. Hence, the amplification factor is given by

$$Q(h\lambda) = 1 \pm hi. \tag{12}$$

Then, it is immediately clear that $|Q(h\lambda)| > 1$ for all $h > 0$. Hence, we conclude that the forward Euler method is never stable if $\varepsilon = 0$.

d From Assignment 1.c., we know that if $\varepsilon = 0$, the eigenvalues of the system are purely imaginary. This implies that the system is analytically (zero) stable if $\varepsilon = 0$.

Nonzero values of ε give the following system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & \varepsilon \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{13}$$

then we get the following eigenvalues $\lambda_{1,2} = \frac{\varepsilon}{2} \pm \frac{1}{2}\sqrt{\varepsilon^2 - 4}$ (real-valued), if $\varepsilon^2 - 4 \geq 0$ and $\lambda = \frac{\varepsilon}{2} \pm \frac{i}{2}\sqrt{4 - \varepsilon^2}$ (nonreal-valued) if $\varepsilon^2 - 4 < 0$. Hence, we consider two cases: real-valued and nonreal-valued eigenvalues.

Real-valued eigenvalues
In this case $|\varepsilon| \geq 2$, and $0 \leq \varepsilon^2 - 4 < \varepsilon^2$, and hence the real-valued eigenvalues have the same sign, which is determined by the sign of ε. Hence, if $\varepsilon \leq -2$, then, the system is stable. Furthermore, if $\varepsilon \geq 2$, then, the system is unstable.

Nonreal-valued eigenvalues
In this case $|\varepsilon| < 2$. The system is analytically unstable if and only if the real part of the eigenvalues is positive. Further, the real part of the eigenvalues is positive if and only if $\varepsilon > 0$. Hence, the system is analytically unstable if and only if $\varepsilon > 0$. Hence, the system is stable if and only if $(-2 <)\varepsilon \leq 0$.

From these arguments, it follows that the system is stable if and only if $\varepsilon \leq 0$.

e Since currently the discriminant, $\varepsilon^2 - 4$, is negative, the eigenvalues are nonreal. Substitution into the amplification factor yields

$$Q(h\lambda) = 1 + \frac{\varepsilon}{2}h \pm \frac{ih}{2}\sqrt{4-\varepsilon^2}. \tag{14}$$

Hence, numerical stability is warranted if

$$|Q(h\lambda)|^2 = (1 + \frac{\varepsilon}{2}h)^2 + \frac{h^2}{4}(4-\varepsilon^2) \leq 1. \tag{15}$$

Hence for stability, we have

$$1 + \varepsilon h + \frac{\varepsilon^2 h^2}{4} + h^2 - \frac{\varepsilon^2 h^2}{4} = 1 + h\varepsilon + h^2 \leq 1. \tag{16}$$

Since $h > 0$, we obtain the following stability criterion

$$h \leq -\varepsilon = |\varepsilon|. \tag{17}$$

If $\varepsilon = -2$, then both eigenvalues are real-valued and given by $\lambda_{1,2} = -1$. For this case, we obtain $Q(\lambda h) = 1 - h$, and stability is warranted if and only if $-1 \leq Q(h\lambda) \leq 1$, hence $h \leq 2(= |\varepsilon|$.

We conclude that for $-2 \leq \varepsilon < 0$, we have a numerically stable solution if and only if $h \leq |\varepsilon|$.

2. a After discretization by the use of finite differences one obtains

$$\frac{-w_{i-1} + 2w_i - w_{i+1}}{h^2} + x_i^2 w_i = x_i. \tag{18}$$

The truncation error is defined by

$$e_i = \frac{-y_{i-1} + 2y_i - y_{i+1}}{h^2} + x_i^2 y_i - x_i. \tag{19}$$

Taylor series of y_{i-1} and y_{i+1} around x_i, gives

$$\begin{aligned} y_{i+1} &= y_i + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y'''(x_i) + \frac{h^4}{4!}y''''(x_i) + O(h^5), \\ y_{i-1} &= y_i - hy'(x_i) + \frac{h^2}{2!}y''(x_i) - \frac{h^3}{3!}y'''(x_i) + \frac{h^4}{4!}y''''(x_i) - O(h^5), \end{aligned} \tag{20}$$

Substitution of the above expressions into the definition of the truncation error gives

$$\varepsilon_i = -y''(x_i) + O(h^2) + x_i^2 y(x_i) - x_i. \tag{21}$$

Using the differential equation $-y'' + x^2 y = x$ finally gives

$$\varepsilon_i = O(h^2). \tag{22}$$

b For this case we have $h = 0.25$, for the points $j \in \{1, 2, 3\}$, the discretization with $w_0 = 0$ and $w_4 = 1$:

$$\begin{aligned}
&32w_1 - 16w_2 + \frac{1}{16}w_1 = \frac{1}{4},\\
&-16w_1 + 32w_2 - 16w_3 + \frac{1}{4}w_2 = \frac{1}{2},\\
&-16w_2 + 32w_3 + \frac{9}{16}w_3 = \frac{3}{4} + 16.
\end{aligned} \tag{23}$$

Hence in matrix-vector form:

$$\begin{pmatrix} 32.0625 & -16 & 0 \\ -16 & 32.25 & -16 \\ 0 & -16 & 32.5625 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.5 \\ 16.75 \end{pmatrix} \tag{24}$$

c The iteration process is a fixed point method. If the process converges we have: $\lim_{n\to\infty} x_n = p$. Using this in the iteration process yields:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} [x_n + h(x_n)(x_n^2 - 4)]$$

Since h is a continuous function one obtains:

$$p = p + h(p)(p^2 - 4)$$

so

$$h(p)(p^2 - 4) = 0.$$

Since $h(x) \neq 0$ for each $x \neq 0$ it follows that $p^2 - 4 = 0$ and thus there are two limits $p = -2$ and $p = 2$.

d The convergence of a fixed point method $x_{n+1} = g(x_n)$ is determined by $g'(p)$. If $|g'(p)| < 1$ the method converges, whereas if $|g'(p)| > 1$ the method diverges if $p_0 \neq p$. For all choices we compute the first derivative in p. For the first method we elaborate all steps. For the other methods we only give the final result. For h_1 we have $g_1(x) = x - \frac{1}{2}x(x^2 - 4) = 3x - \frac{1}{2}x^3$. The first derivative is:

$$g_1'(x) = 3 - \frac{3}{2}x^2$$

Substitution of $p = 2$ yields:

$$g_1'(2) = 3 - \frac{3}{2}4 = 3 - 6 = -3.$$

Since $|g_1'(2)| = 3 > 1$ this method is divergent.

For the second method we have:

$$g_2(x) = x - \frac{1}{3}(x^2 - 4)$$

$$g_2'(x) = 1 - \frac{2}{3}x$$

Since $|g_2'(2)| = |-\frac{1}{3}| = \frac{1}{3} < 1$ the method converges with convergence factor $\frac{1}{3}$.

For the third method we have:

$$g_3(x) = x - \frac{1}{2x}(x^2 - 4) = \frac{x}{2} + \frac{2}{x}$$

$$g_3'(x) = \frac{1}{2} - \frac{2}{x^2}$$

Note that $g_3'(2) = \frac{1}{2} - \frac{2}{4} = 0$ the method is convergent with convergence factor 0.

Concluding we note that the third method is the fastest since $|g_3'(2)| < |g_2'(2)|$.

e We use the iteration process:

$$x_{n+1} = x_n - \frac{1}{3}(x_n^2 - 4)$$

Starting from $x_0 = 3$ we obtain the following iterates:

$$x_1 = 1.3333$$

$$x_2 = 2.0741$$

$$x_3 = 1.9735$$

Note that the method indeed converges and that the convergence is alternating.

f To estimate the error in p we first approximate the function f in the neighboorhood of p by the first order Taylor polynomial:

$$P_1(x) = f(p) + (x - p)f'(p) = (x - p)f'(p).$$

Due to the measurement errors we know that

$$(x - p)f'(p) - \epsilon_{max} \leq \hat{P}_1(x) \leq (x - p)f'(p) + \epsilon_{max}.$$

This implies that the perturbed root $\hat{p}$ is bounded by the roots of $(x-p)f'(p) - \epsilon_{max}$ and $(x - p)f'(p) + \epsilon_{max}$, which leads to

$$p - \frac{\epsilon_{max}}{|f'(p)|} \leq \hat{p} \leq p + \frac{\epsilon_{max}}{|f'(p)|}.$$