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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Thursday January 20 2011, 18:30-21:30

1. (a) The local truncation error is given by

$$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h}, \tag{1}$$

in which we determine y_{n+1} by the use of Taylor expansions around t_n:

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2} y''(t_n) + O(h^3). \tag{2}$$

We bear in mind that

$$\begin{aligned} y'(t_n) &= f(t_n, y_n) \\ y''(t_n) &= \frac{df(t_n, y_n)}{dt} = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} y'(t_n) = \\ &\frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} f(t_n, y_n). \end{aligned} \tag{3}$$

Hence

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2} \left(\frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} f(t_n, y_n) \right) + O(h^3). \tag{4}$$

After substitution of the predictor $z^*_{n+1} = y_n + hf(t_n, y_n)$ into the corrector, and after using a Taylor expansion around (t_n, y_n), we obtain for z_{n+1}

$$\begin{aligned} z_{n+1} &= y_n + \tfrac{h}{2} \left(f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n)) \right) = \\ y_n &+ \frac{h}{2} \left(f(t_n, y_n) + f(t_n, y_n) + h\left(\frac{\partial f(t_n, y_n)}{\partial t} + f(t_n, y_n) \frac{\partial f(t_n, y_n)}{\partial y}\right) + O(h^2) \right). \end{aligned} \tag{5}$$

Herewith, one obtains

$$y_{n+1} - z_{n+1} = O(h^3), \text{ and hence } \tau_{n+1}(h) = \frac{O(h^3)}{h} = O(h^2). \tag{6}$$

(b) Let $x_1 = y$ and $x_2 = y'$, then $y'' = x_2'$, and hence

$$\begin{aligned} x_2' + 4x_2 + 3x_1 &= \cos(t), \\ x_2 &= x_1'. \end{aligned} \tag{7}$$

We write this as

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -3x_1 - 4x_2 + \cos(t). \end{aligned} \tag{8}$$

Finally, this is represented in the following matrix-vector form:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos(t) \end{pmatrix}. \tag{9}$$

In which, we have the following matrix $A = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}$ and $f = \begin{pmatrix} 0 \\ \cos(t) \end{pmatrix}$.

The initial conditions are defined by $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(c) Application of the Modified Euler method to the system $\underline{x}' = A\underline{x} + \underline{f}$, gives

$$\begin{aligned} \underline{w}_1^* &= \underline{w}_0 + h\left(A\underline{w}_0 + \underline{f}_0\right), \\ \underline{w}_1 &= \underline{w}_0 + \tfrac{h}{2}\left(A\underline{w}_0 + f_0 + A\underline{w}_1^* + \underline{f}_1\right). \end{aligned} \tag{10}$$

With the initial condition $\underline{w}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $h = 0.1$, this gives the following result for the predictor

$$\underline{w}_1^* = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{10}\left(\begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 6/5 \\ 1 \end{pmatrix}. \tag{11}$$

The corrector is calculated as follows

$$\underline{w}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \tfrac{1}{20}\left(\begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} 6/5 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos(\frac{1}{10}) \end{pmatrix}\right) =$$

$$= \begin{pmatrix} 1.1500 \\ 1.1698 \end{pmatrix} \tag{12}$$

(d) Consider the test equation $y' = \lambda y$, then one gets

$$\begin{aligned} w_{n+1}^* &= w_n + h\lambda w_n = (1 + h\lambda)w_n, \\ w_{n+1} &= w_n + \frac{h}{2}(\lambda w_n + \lambda w_{n+1}^*) = \\ &= w_n + \frac{h}{2}(\lambda w_n + \lambda(w_n + h\lambda w_n)) = (1 + h\lambda + \frac{(h\lambda)^2}{2})w_n. \end{aligned} \tag{13}$$

Hence the amplification factor is given by

$$Q(h\lambda) = 1 + h\lambda + \frac{(h\lambda)^2}{2}. \tag{14}$$

(e) First, we determine the eigenvalues of the matrix A. Subsequently, the eigenvalues are substituted into the amplification factor. The eigenvalues of the matrix A are given by $\lambda_1 = -1$ and $\lambda_2 = -3$. We first check the amplification factor of $\lambda_1 = -1$:

$$-1 \leq 1 - h + \frac{1}{2}h^2 \leq 1 \tag{15}$$

The first inequality leads to

$$0 \leq 2 - h + \frac{1}{2}h^2$$

Since the discriminant of this equation is equal to $1-4*\frac{1}{2}*2 = -3$ the inequality always holds. The second inequality leads to

$$-h + \frac{1}{2}h^2 \leq 0$$

so

$$\frac{1}{2}h^2 \leq h$$

which implies

$$h \leq 2$$

Now we check the amplification factor of $\lambda_2 = -3$:

$$-1 \leq 1 - 3h + \frac{1}{2}9h^2 \leq 1 \tag{16}$$

The first inequality leads to

$$0 \leq 2 - 3h + \frac{1}{2}9h^2$$

Since the discriminant of this equation is equal to $9 - 4 * \frac{9}{2} * 2 = -27$ the inequality always holds. The second inequality leads to

$$-3h + \frac{9}{2}h^2 \leq 0$$

so

$$\frac{3}{2}h^2 \leq h$$

which implies

$$h \leq \frac{2}{3}$$

So the modified Euler method is stable if $h \leq \frac{2}{3}$.

2. (a) We have to check whether the requirements for the Convergence Theorem (see Theorem 4.3.2 in Vuik et al.) on convergence are satisfied. We have to remark that these requirements give a sufficient condition for convergence to the fixed point if we choose an initial value in a neighborhood around the fixed point p. The theorem is formulated as follows:

Theorem: *If there exists a $\delta > 0$ such that $g(x) \in C^1[p-\delta, p+\delta]$ (the first order derivative of $g(x)$ is continuous), then, the fixed point method converges for each initial value $p_0 \in [p-\delta, p+\delta]$ if the following hypotheses are satisfied:*

i. $g : [p-\delta, p+\delta] \longrightarrow [p-\delta, p+\delta]$;

ii. There exists a $r > 0$ such that

$$|g'(x)| \leq r < 1, \text{ for } x \in [p-\delta, p+\delta].$$

Firstly, the derivative of $g(x)$ is given by

$$g'(x) = 1 - \frac{f'(x)}{\alpha}.$$

Further, we have

$$g'(p) = 1 - \frac{f'(p)}{\alpha} > 1 - \frac{f'(p)}{f'(p)} = 0,$$

since $0 < f'(p) < \alpha$. From this, it also follows that

$$g'(p) = 1 - \frac{f'(p)}{\alpha} < 1,$$

since $f'(p) > 0$ and $\alpha > 0$. When we combine these bounds for the derivative of g' with continuity of $f'(x)$, and hence also with continuity of $g'(x)$ around p, it follows that there is a neighborhood around p for which we have $0 < g'(x) < 1$. In other words, mathematically speaking: There exists a $\delta > 0$ for which $0 < g'(x) < 1$ for each $x \in [p-\delta, p+\delta]$. Hence the first hypothesis of the convergence theorem is satisfied.

Further, via the Mean Value Theorem, we get

$$g(p+\delta) = g(p) + \delta g'(\xi_1), \text{ for a } \xi_1 \in (p-\delta, p+\delta) \text{ and,}$$

$$g(p-\delta) = g(p) - \delta g'(\xi_2), \text{ for a } \xi_2 \in (p-\delta, p+\delta).$$

This yields with $0 < g'(\xi) < 1$ and $g(p) = p$:

$$g(p-\delta) = g(p) - \delta g'(\xi_1) > p - \delta, \text{ and } g(p+\delta) = g(p) + \delta g'(\xi_2) < p + \delta.$$

Hence, we have $g(p \pm \delta) \in [p-\delta, p+\delta]$. Since $g(x)$ is monotonical on $[p-\delta, p+\delta]$, $g(x)$ cannot assume any values outside the range $[p-\delta, p+\delta]$ if $x \in [p-\delta, p+\delta]$. Hence, we have

$$g(x) \in [p-\delta, p+\delta], \text{ for } x \in [p-\delta, p+\delta],$$

which is equivalent to the second hypothesis. This all sustains convergence if the initial guess is chosen within a neighborhood around the fixed point p.

(b) The method of Newton-Raphson is based on linearization around the iterate p_n. This is given by

$$L(x) = f(p_n) + (x - p_n)f'(p_n). \tag{17}$$

Next, we determine p_{n+1} such that $L(p_{n+1}) = 0$, that is

$$f(p_n) + (p_{n+1} - p_n)f'(p_n) = 0 \Leftrightarrow p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \qquad f'(p_n) \neq 0. \tag{18}$$

This result can also be proved graphically, see book, chapter 4.

(c) We have $f(x) = x^2 - 2x - 2$, so $f'(x) = 2x - 2$ and hence

$$p_{n+1} = p_n - \frac{p_n^2 - 2p_n - 2}{2p_n - 2}.$$

With the initial value $p_0 = 2$, this gives

$$p_1 = 2 - \frac{4 - 4 - 2}{4 - 2} = 3.$$

(d) We have $f'(x) = 2x - 2$ and hence $f'(1) = 0$ with starting value $p_0 = 1$. In the recursion, one divides by zero. Division by zero does not make any sense, so $p_0 = 1$ is not a suitable starting value. Geometrically, one may remark that the tangent is horizontal on $p_0 = 1$.

(e) We answer both questions sequentially:

- The linear interpolation polynomial with points x_0 en x_1 is given by:

$$P_1(x) = y(x_0)\frac{x - x_1}{x_0 - x_1} + y(x_1)\frac{x - x_0}{x_1 - x_0} = -(x-1) + 3(x-2/3) = 2x - 1. \tag{19}$$

- We have $P_1(x) = 1/2 \Leftrightarrow 2x - 1 = 1/2$. Solution of this equation in x gives $x = \frac{3}{4}$.