DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Thursday August 25 2011, 18:30-21:30

1. (a) The local truncation error is given by

$$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h}. \tag{1}$$

Here we obtain y_{n+1} by a Taylor expansion around t_n:

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2}y''(t_n) + O(h^3). \tag{2}$$

For z_{n+1}, we obtain, after substitution of the predictor step for z^*_{n+1} into the corrector step

$$z_{n+1} = y_n + h\left((1-\theta)f(t_n, y_n) + \theta f(t_n + h, y_n + hf(t_n, y_n))\right) \tag{3}$$

After a Taylor expansion of $f(t_n+h, y_n+hf(t_n, y_n))$ around (t_n, y_n) one obtains:

$$z_{n+1} = y_n + h\left((1-\theta)f(t_n, y_n) + \theta(f(t_n, y_n) + h(\frac{\partial f(t_n, y_n)}{\partial t} + f(t_n, y_n)\frac{\partial f(t_n, y_n)}{\partial y})) + O(h^2)\right). \tag{4}$$

From the differential equation we know that:

$$y'(t_n) = f(t_n, y_n) \tag{5}$$

From the Chain Rule of Differentiation, we derive

$$y''(t_n) = \frac{df(t_n, y_n)}{dt} = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}y'(t_n) \tag{6}$$

after substitution of the differential equation one obtains:

$$y''(t_n) = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}f(t_n, y_n) \tag{7}$$

This implies that $z_{n+1} = y_n + hy'(t_n) + \theta h^2 y''(t_n)$. Subsequently, it follows that

$$y_{n+1} - z_{n+1} = O(h^2), \text{ and, hence } \tau_{n+1}(h) = \frac{O(h^2)}{h} = O(h) \text{ for } 0 \le \theta \le 1, \tag{8}$$

$$y_{n+1} - z_{n+1} = O(h^3), \text{ and, hence } \tau_{n+1}(h) = \frac{O(h^3)}{h} = O(h^2) \text{ for } \theta = \frac{1}{2}. \tag{9}$$

(b) Consider the test equation $y' = \lambda y$, then, herewith, one obtains

$$\begin{aligned} &w_{n+1}^* = w_n + h\lambda w_n = (1+h\lambda)w_n, \\ &w_{n+1} = w_n + h((1-\theta)\lambda w_n + \theta\lambda w_{n+1}^*) = \\ &= w_n + h((1-\theta)\lambda w_n + \theta\lambda(w_n + h\lambda w_n)) = (1 + h\lambda + \theta(h\lambda)^2)w_n. \end{aligned} \tag{10}$$

Hence the amplification factor is given by

$$Q(h\lambda) = 1 + h\lambda + \theta(h\lambda)^2. \tag{11}$$

(c) We start this exercise by using the following vector:

$$x_1 = y$$

$$x_2 = y'$$

From this it follows that

$$x_1' = y' = x_2$$
$$x_2' = y'' = -4y' - 8y + t^2 - 1 = -4x_2 - 8x_1 + t^2 - 1$$

where we have used the second order differential equation. We can write this as follows in matrix-vector notation:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -8 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ t^2 - 1 \end{pmatrix}$$

So it follows that $A = \begin{pmatrix} 0 & 1 \\ -8 & -4 \end{pmatrix}$ and $f(t) = 0$ and $g(t) = t^2 - 1$.

(d) In order to do one step we first note that

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The predictor step with $h = 1$ now gives:

$$w_1^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1\left(\begin{pmatrix} 0 & 1 \\ -8 & -4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

Finally the correction step with $\theta = \frac{1}{2}$ leads to

$$w_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -5 \end{pmatrix} + \frac{1}{2}\left(\begin{pmatrix} 0 & 1 \\ -8 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -1.5 \\ 2.5 \end{pmatrix}$$

(e) Compute the eigenvalues of matrix $\begin{pmatrix} 0 & 1 \\ -8 & -4 \end{pmatrix}$. To do this we compute the determinant of $\begin{pmatrix} -\lambda & 1 \\ -8 & -4-\lambda \end{pmatrix}$, which is equal to $\lambda^2+4\lambda+8$. The roots of this polynomial are equal to $\lambda_1 = -2+2i$ and $\lambda_2 = -2-2i$. Since $\lambda_2 = \bar{\lambda_1}$ it is sufficient to consider λ_1 only. For $h = 1$ we obtain $h\lambda_1 = -2+2i$. This implies that

$$Q(h\lambda_1) = 1 + h\lambda_1 + \theta(h\lambda_1)^2$$
$$Q(h\lambda_1) = 1 + (-2+2i) + \theta(-2+2i)^2$$
$$Q(h\lambda_1) = 1 - 2 + 2i + \theta(4 - 8i - 4) = -1 + i(2 - 8\theta)$$

In order to check that $|Q(h\lambda_1)| \le 1$, we compute the modulus of $Q(h\lambda_1)$, which is equal to

$$\sqrt{1^2 + (2-8\theta)^2}$$

It is easy to see that this is only less than or equal to 1 if $\theta = \frac{1}{4}$.

2. (a) The Taylor polynomials around 0 are given by:

$$\begin{aligned} f(0) &= f(0)\ , \\ f(h) &= f(0) + hf'(0) + \frac{h^2}{2}f''(0) + \frac{h^3}{6}f'''(\xi_1)\ , \\ f(2h) &= f(0) + 2hf'(0) + 2h^2f''(0) + \frac{(2h)^3}{6}f'''(\xi_2)\ . \end{aligned}$$

Here $\xi_1 \in (0,h)$, $\xi_2 \in (0,2h)$. We know that $Q(h) = \frac{\alpha_0}{h^2}f(0) + \frac{\alpha_1}{h^2}f(h) + \frac{\alpha_2}{h^2}f(2h)$, which should be equal to $f''(0) + O(h)$. This leads to the following conditions:

$$\begin{array}{lcccccc} f(0): & \frac{\alpha_0}{h^2} & + & \frac{\alpha_1}{h^2} & + & \frac{\alpha_2}{h^2} & = 0\ , \\ f'(0): & & & \frac{h\alpha_1}{h^2} & + & \frac{2h\alpha_2}{h^2} & = 0\ , \\ f''(0): & & & \frac{h^2}{2h^2}\alpha_1 & + & \frac{2h^2\alpha_2}{h^2} & = 1\ . \end{array}$$

This can also be written as

$$\begin{array}{lcccccc} f(0): & \alpha_0 & + & \alpha_1 & + & \alpha_2 & = 0\ , \\ f'(0): & & & \alpha_1 & + & 2\alpha_2 & = 0\ , \\ f''(0): & & & \frac{\alpha_1}{2} & + & 2\alpha_2 & = 1\ . \end{array}$$

(b) The truncation error follows from the Taylor polynomials:

$$f''(0) - Q(h) = f''(0) - \frac{f(0) - 2f(h) + f(2h)}{h^2} = \frac{-\frac{2h^3}{6}f'''(\xi_1) + \frac{8h^3}{6}f'''(\xi_2)}{h^2}$$
$$= hf'''(\xi).$$

(c) Note that

$$f''(0) - Q(h) = Kh \tag{12}$$

$$f''(0) - Q(\frac{h}{2}) = K(\frac{h}{2}) \tag{13}$$

Subtraction gives:

$$Q(\frac{h}{2}) - Q(h) = Kh - K\frac{h}{2} = K(\frac{h}{2}). \tag{14}$$

We choose $h = \frac{1}{2}$. Then $Q(h) = Q(\frac{1}{2}) = \frac{0-2\times 0.1250+1}{0.25} = 3$ and $Q(\frac{h}{2}) = Q(\frac{1}{4}) = \frac{0-2\times 0.0156+0.1250}{(\frac{1}{4})^2} = 1.5008$. Combining (13) and (14) shows that

$$f''(0) - Q(\frac{1}{4}) = Q(\frac{1}{4}) - Q(\frac{1}{2}) = -1.4992$$

(d) To estimate the rounding error we note that

$$|Q(h) - \hat{Q}(h)| = |\frac{(f(0) - \hat{f}(0)) - 2(f(h) - \hat{f}(h)) + (f(2h) - \hat{f}(2h))}{h^2}|$$

$$\leq \frac{|f(0) - \hat{f}(0)| + 2|f(h) - \hat{f}(h)| + |f(2h) - \hat{f}(2h)|}{h^2} \leq \frac{4\epsilon}{h^2},$$

so $C_1 = 4$. Since only 4 digits are given the rounding error is: $\epsilon = 0.00005$.

(e) The total error is bounded by

$$|f''(0) - \hat{Q}(h)| = |f''(0) - Q(h) + Q(h) - \hat{Q}(h)|$$

$$\leq |f''(0) - Q(h)| + |Q(h) - \hat{Q}(h)|$$

$$\leq 6h + \frac{4\epsilon}{h^2} = g(h)$$

This is minimal for h_{opt}, for which $g'(h_{opt}) = 0$. Note that $g'(h) = 6 - \frac{8\epsilon}{h^3}$. This implies that $h_{opt}^3 = \frac{4\epsilon}{3}$, so $h_{opt} = (\frac{4\epsilon}{3})^{\frac{1}{3}} \approx 0.0405$.