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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Thursday April 14 2011, 18:30-21:30

1. We consider the following method:

$$y_{n+1} = y_n + h\left(\alpha f(t_n, y_n) + \beta f(t_{n-1}, y_{n-1})\right). \tag{1}$$

(a) The local truncation error is given by

$$\tau_{n+1} = \frac{y_{n+1} - z_{n+1}}{h},$$

where y_{n+1} is the exact solution at time t_{n+1} and z_{n+1} is the numerical method (1) applied to y_{n-1} and y_n. We will need the following Taylor expansions:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}y''_n + \mathcal{O}(h^3),$$

$$y'_{n-1} = f(t_{n-1}, y_{n-1}) = y'_n - hy''_n + \mathcal{O}(h^2).$$

We then have for z_{n+1} :

$$z_{n+1} = y_n + h(\alpha + \beta)y'_n - h^2\beta y''_n + \mathcal{O}(h^3).$$

Subtracting this from y_{n+1} and dividing by h we have the local truncation error is:

$$\tau_{n+1} = (1 - (\alpha + \beta))y'_n + h\left(\frac{1}{2} + \beta\right)y''_n + \mathcal{O}(h^2).$$

Since

$$\alpha = \frac{3}{2}, \qquad \beta = -\frac{1}{2}$$

we obtain $\tau_{n+1} = \mathcal{O}(h^2)$.

(b) Substituting the relation $y_j = [Q(h\lambda)]y_{j-1}$ and the test equation $y' = \lambda y = f(t, y)$ in (1) gives the following:

$$[Q(h\lambda)]^2 y_{n-1} = Q(h\lambda)y_{n-1} + h\left(\frac{3}{2}\lambda Q(h\lambda)y_{n-1} - \frac{1}{2}\lambda y_{n-1}\right).$$

This can be rewritten as

$$[Q(h\lambda)]^2 - Q(h\lambda)\left(1 + \frac{3}{2}h\lambda\right) + \frac{1}{2}h\lambda = 0.$$

We therefore have that

$$Q_1(h\lambda) = \frac{1}{2}\left(1 + \frac{3}{2}h\lambda + \sqrt{\left(\frac{3}{2}h\lambda\right)^2 + h\lambda + 1}\right) \tag{2}$$

$$Q_2(h\lambda) = \frac{1}{2}\left(1 + \frac{3}{2}h\lambda - \sqrt{\left(\frac{3}{2}h\lambda\right)^2 + h\lambda + 1}\right) \tag{3}$$

(c) Since the discriminant of $\left(\frac{3}{2}h\lambda\right)^2 + h\lambda + 1$ is negative the value of $\left(\frac{3}{2}h\lambda\right)^2 + h\lambda + 1$ is always positive, so both $Q_1(h\lambda)$ and $Q_2(h\lambda)$ are real numbers. This implies that we must have $-1 \le Q_2(h\lambda) < Q_1(h\lambda) \le 1$. Note that $Q_1(h\lambda) \le 1$ is satisfied for all h because it simplifies to

$$\frac{1}{2}\left(1 + \frac{3}{2}h\lambda + \sqrt{\left(\frac{3}{2}h\lambda\right)^2 + h\lambda + 1}\right) \le 1$$

$$\sqrt{\left(\frac{3}{2}h\lambda\right)^2 + h\lambda + 1} \le 2 - 1 - \frac{3}{2}h\lambda$$

$$\sqrt{\left(\frac{3}{2}h\lambda\right)^2 + h\lambda + 1} \le 1 - \frac{3}{2}h\lambda$$

Squaring both sides gives

$$\left(\frac{3}{2}h\lambda\right)^2 + h\lambda + 1 \le 1 - 3h\lambda + \left(\frac{3}{2}h\lambda\right)^2,$$

which implies

$$0 \le -4h\lambda$$

which is always true for negative real values of λ.

For $-1 \le Q_2(h\lambda)$, we can write this as

$$\begin{aligned} -2 &\le 1 + \frac{3}{2}h\lambda - \sqrt{\left(\frac{3}{2}h\lambda\right)^2 + h\lambda + 1} \\ (3 + \frac{3}{2}h\lambda)^2 &\ge \left(\frac{3}{2}h\lambda\right)^2 + h\lambda + 1 \\ \left(\frac{3}{2}h\lambda\right)^2 + 9h\lambda + 9 &\ge \left(\frac{3}{2}h\lambda\right)^2 + h\lambda + 1 \end{aligned} \tag{4}$$

which simplifies to

$$h \leq -\frac{1}{\lambda}.$$

Consider the system

$$\mathbf{y}' = \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ -\cos(t) \end{bmatrix}. \tag{5}$$

(d) The eigenvalues of the matrix in (5) are given by

$$det(A - \lambda I) = (-4-\lambda)^2 - 1 = \lambda^2 + 8\lambda + 15 = 0.$$

This gives the values $\lambda_1 = -3$ and $\lambda_2 = -5$. Therefore, in order to apply our method to the system (5), we have the stability criteria

$$h \leq \frac{1}{3} \text{ and } h \leq \frac{1}{5}.$$

Since the strongest condition should be satisfied the method is stable for

$$h \leq \frac{1}{5}.$$

(e) Method (1) converges as long as $h < \frac{1}{\max_\lambda |\lambda|}$ because a stable and consistent scheme converges (Lax equivalence theorem).

2. (a) The linear Lagrangian interpolatory polynomial, with nodes x_0 and x_1, is given by

$$p_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1). \tag{6}$$

This is evident from application of the given formula.

(b) The quadratic Lagrangian interpolatory polynomial with nodes x_0, x_1 and x_2 is given by

$$p_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2). \tag{7}$$

This is also evident from application of the given formula.

(c) To this extent, we compute $p_1(0.5)$ and $p_2(0.5)$ for both linear and quadratic Lagrangian interpolation as approximations at $x = 0.5$. For linear interpolation, we have

$$p_1(0.5) = 0.5 + \frac{1}{2} \cdot 2 = \frac{3}{2}, \tag{8}$$

and for quadratic interpolation, one obtains

$$p_2(0.5) = \frac{(0.5-1)(0.5-2)}{1 \cdot (-2)} \cdot 1 + \frac{(0.5-0)(0.5-2)}{1 \cdot (-1)} \cdot 2 + \frac{(0.5-0)(0.5-1)}{2 \cdot 1} \cdot 4 = \frac{11}{8} = 1.375. \tag{9}$$

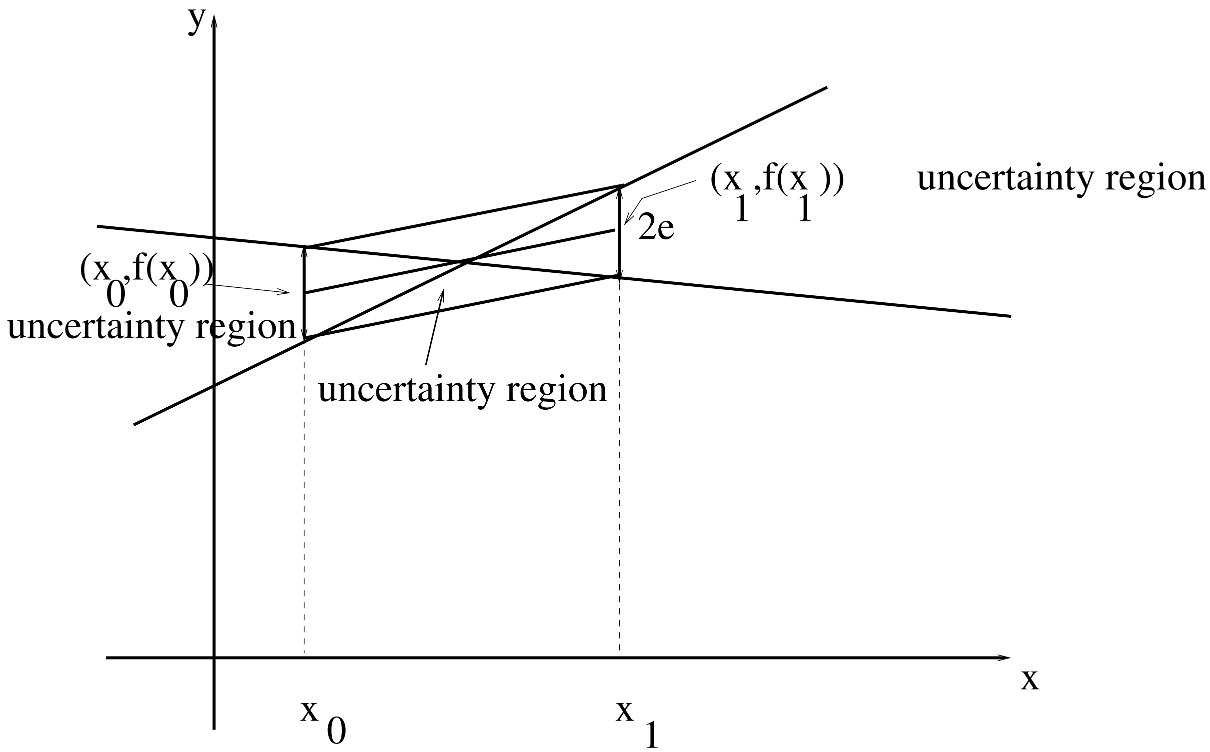

Figure 1: The measured values and the error using linear interpolation.

(d) Consider Figure 1. For interpolation, the error is bounded and for extrapolation, the error may become arbitrarily large as we move more and more outside the interval of the measured values. Of course, also a more algebraic motivation is allowed. We note that this effect may become worse if a higher order interpolatory formula is used.

(e) i We integrate $f(x)$, in which we approximate $f(x)$ by $p_1(x)$, then it follows:

$$\int_{x_0}^{x_1} f(x)dx \approx \int_{x_0}^{x_1} p_1(x)dx = \int_{x_0}^{x_1} \left\{ f(x_0)\frac{x-x_1}{x_0-x_1} + f(x_1)\frac{x-x_0}{x_1-x_0} \right\} dx =$$

$$= \left[\frac{1}{2}\frac{(x-x_0)^2}{x_1-x_0} f(x_1) \right]_{x_0}^{x_1} + \left[\frac{1}{2}\frac{(x-x_1)^2}{x_0-x_1} f(x_0) \right]_{x_0}^{x_1} = \frac{1}{2}(x_1-x_0)(f(x_0)+f(x_1)). \tag{10}$$

This is the Trapezoidal Rule.

ii The magnitude of the error of the numerical integration over interval $[x_0, x_1]$

is given by

$$
\begin{aligned}
&|\int_{x_0}^{x_1} f(x)dx - \int_{x_0}^{x_1} p_1(x)dx| = |\int_{x_0}^{x_1} \left(f(x) - p_1(x)\right) dx| = \\
&|\int_{x_0}^{x_1} \frac{1}{2}(x-x_0)(x-x_1)f''(\chi(x))dx| \leq \frac{1}{2} \max_{x\in[x_0,x_1]} |f''(x)| \int_{x_0}^{x_1} (x-x_0)(x-x_1)dx = \\
&\frac{1}{12}(x_1-x_0)^3 \max_{x\in[x_0,x_1]} |f''(x)|.
\end{aligned}
\tag{11}
$$