DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)

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1. a The local truncation error is defined by

$$\tau_{n+1}(h) := \frac{y_{n+1} - z_{n+1}}{h}, \tag{1}$$

where $y_n := y(t_n)$ represents the exact solution and

$$z_{n+1} = y_n + hf(t_{n+1}, z_{n+1}), \tag{2}$$

represents the approximation of the numerical solution at t_{n+1} upon using y_n for the previous time step. Since, we use the test equation $y' = \lambda y$, we express y_{n+1} in terms of y_n as follows

$$y_{n+1} = y_n e^{\lambda h} = y_n(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + O(h^3)). \tag{3}$$

From (2), we use the test equation and the geometric series

$$z_{n+1} = \frac{y_n}{1 - h\lambda} = y_n(1 + h\lambda + h^2\lambda^2 + O(h^3)). \tag{4}$$

Substitution of equations (3) and (4) into the definition of the local truncation error, gives

$$\tau_{n+1}(h) = \frac{y_n}{h}\left[\frac{h^2\lambda^2}{2} + O(h^3)\right] = O(h). \tag{5}$$

b Using the test equation, we get

$$w_{n+1} = w_n + h\lambda w_{n+1}, \tag{6}$$

where w_n denotes the numerical approximation of y_n. The above equation implies

$$w_{n+1} = \frac{w_n}{1 - h\lambda} =: Q(h\lambda)w_n. \tag{7}$$

Here $Q(h\lambda)$ represents the amplification factor. For numerical stability, we require the modulus of the amplification factor to satisfy

$$Q(h\lambda) \leq 1, \text{ hence } |\frac{1}{1 - h\lambda}| = \frac{1}{|1 - h\lambda|} \leq 1. \tag{8}$$

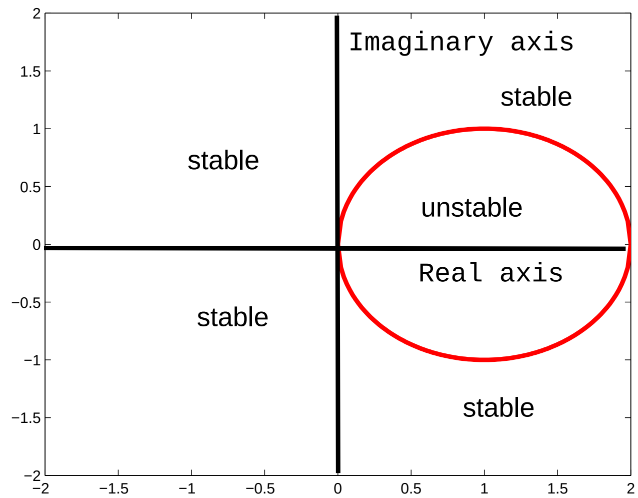

Figure 1: The region of stability of the backward Euler method.

From the above equation, it is clear that

$$|1 - h\lambda| \geq 1, \tag{9}$$

and with $\lambda = \mu + i\nu$, we get

$$(1 - h\mu)^2 + (h\nu)^2 \geq 1. \tag{10}$$

This area is the whole complex plane except the unit circle with center $(1, 0)$, see Figure 1.

c Consider the equations that we have to solve

$$\begin{aligned} y_1' &= y_1(1 - (y_1 + 3y_2)) =: f_1(y_1, y_2), \\ y_2' &= y_2(1 - (y_1 + y_2)) =: f_2(y_1, y_2), \end{aligned} \tag{11}$$

Then, the Jacobi matrix is given by

$$J(y_1, y_2) := \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(y_1, y_2) & \frac{\partial f_1}{\partial y_2}(y_1, y_2) \\ \frac{\partial f_2}{\partial y_1}(y_1, y_2) & \frac{\partial f_2}{\partial y_2}(y_1, y_2) \end{pmatrix} = \begin{pmatrix} 1 - 2y_1 - 3y_2 & -3y_1 \\ -y_2 & 1 - (y_1 + 2y_2) \end{pmatrix}. \tag{12}$$

For the equilibrium $(0, 1)$, we have

$$J(y_1, y_2) := \begin{pmatrix} -2 & 0 \\ -1 & -1 \end{pmatrix}. \tag{13}$$

Hence the eigenvalues are given by $\lambda_1 = -2$ and $\lambda_2 = -1$.

d - We have $\lambda_1 = -2$ and $\lambda_2 = -1$, hence with $h > 0$, this implies that $h\lambda < 0$ (thus real-valued), then from Figure 1, it is clear that the backward Euler is stable for any $h > 0$.

- Since the eigenvalues are real-valued and negative, we use

$$h|\lambda| \leq 2, \tag{14}$$

as stability bound for the forward Euler method. With $\lambda_1 = -2$ and $\lambda_2 = -1$, we get $h \leq 1$ as the maximum allowable time step to warrant numerical stability, based on linear stability analysis around $(0, 1)$.

2. (a) The first order backward difference formula for the first derivative is given by

$$f'(t) \approx \frac{f(t) - f(t-h)}{h}.$$

Using $t = 2$, and $h = 1$ the approximation of the velocity is

$$\frac{f(2) - f(1)}{1} = 250 - 215 = 35 \text{ (m/s)}.$$

(b) Taylor polynomials are:

$$\begin{aligned}
f(0) &= f(2h) - 2hf^{'}(2h) + 2h^2 f^{''}(2h) - \frac{(2h)^3}{6} f^{'''}(\xi_0)\,, \\
f(h) &= f(2h) - hf^{'}(2h) + \frac{h^2}{2} f^{''}(2h) - \frac{h^3}{6} f^{'''}(\xi_1)\,, \\
f(2h) &= f(2h).
\end{aligned}$$

We know that $Q(h) = \frac{\alpha_0}{h} f(0) + \frac{\alpha_1}{h} f(h) + \frac{\alpha_2}{h} f(2h)$, which should be equal to $f'(2h) + O(h^2)$. This leads to the following conditions:

$$\begin{array}{rcrcrcl}
\frac{\alpha_0}{h} & + & \frac{\alpha_1}{h} & + & \frac{\alpha_2}{h} & = & 0\,, \\
-2\alpha_0 & - & \alpha_1 & & & = & 1\,, \\
2\alpha_0 h & + & \frac{1}{2}\alpha_1 h & & & = & 0\,.
\end{array}$$

(c) The truncation error follows from the Taylor polynomials:

$$f'(2h) - Q(h) = f'(2h) - \frac{f(0) - 4f(h) + 3f(2h)}{2h} = \frac{\frac{8h^3}{6} f^{'''}(\xi_0) - 4(\frac{h^3}{6} f^{'''}(\xi_1))}{2h} = \frac{1}{3} h^2 f'''(\xi).$$

Using the new formula with $h = 1$ we obtain the estimate:

$$\frac{f(0) - 4f(1) + 3f(2)}{2} = \frac{200 - 4 \times 215 + 3 \times 250}{2} = 45 \text{ (m/s)}.$$

Note that the estimated velocity of the vehicle is larger than the maximum speed of 40 (m/s).

(d) To estimate the measuring error we note that

$$|Q(h) - \hat{Q}(h)| = |\frac{(f(0) - \hat{f}(0)) - 4(f(h) - \hat{f}(h)) + 3(f(2h) - \hat{f}(2h))}{2h}|$$

$$\leq \frac{|f(0) - \hat{f}(0)| + 4|f(h) - \hat{f}(h)| + 3|f(2h) - \hat{f}(2h)|}{2h} = \frac{4\epsilon}{h},$$

so $C_1 = 4$.

(e) The measuring error is less than 1.25 (m). This implies that $\epsilon = 1.25$. Putting this in the estimate for the measuring error leads to

$$|Q(h) - \hat{Q}(h)| \leq 4 \times 1.25 = 5 \text{ (m/s)}.$$

Since the difference between the estimate 45 m/s and the maximum speed 40 m/s is less than 5 m/s it is possible that the actual velocity of the vehicle is equal to the maximum speed, due to the measuring errors.

(f) A first order backward estimate of the second derivative $f''(2h)$ is given by

$$\frac{f(0) - 2f(h) + f(2h)}{h^2}.$$

To check this one can use the same system as in part (b), where one takes as coefficients $\frac{\alpha_i}{h^2}$, the right-hand sides of the first and second equations are 0 and the right-hand side of the third equation is 1. Using $h = 1$ the estimate is equal to

$$\frac{200 - 2 \times 215 + 250}{1} = 20 \text{ (m/s}^2\text{)}.$$

To estimate the measuring error we note that

$$|\frac{(f(0) - \hat{f}(0)) - 2(f(h) - \hat{f}(h)) + (f(2h) - \hat{f}(2h))}{h^2}|$$

$$\leq \frac{|f(0) - \hat{f}(0)| + 2|f(h) - \hat{f}(h)| + |f(2h) - \hat{f}(2h)|}{h^2} = \frac{4\epsilon}{h^2}.$$

Using $\epsilon = 1.25$ shows that the measuring error in the estimate of second derivative is less than 5 (m/s^2). Assuming that the truncation is less than the measuring error we conclude that the second derivative is positive, which implies that the vehicle accelerates and thus its velocity will become larger than the maximum speed for $t > 2$.