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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Thursday April 15 2010, 18:30-21:30

1. (a) Replace $f(t, y)$ by λy in the RK$_4$ formulas:

$$\begin{aligned}
k_1 &= h\lambda w_n \\
k_2 &= h\lambda(w_n + \tfrac{1}{2}k_1) = h\lambda(1 + \tfrac{1}{2}h\lambda)w_n \\
k_3 &= h\lambda(w_n + \tfrac{1}{2}k_2) = h\lambda(1 + \tfrac{1}{2}h\lambda(1 + \tfrac{1}{2}h\lambda))w_n \\
k_4 &= h\lambda(w_n + k_3) = h\lambda(1 + h\lambda(1 + \tfrac{1}{2}h\lambda(1 + \tfrac{1}{2}h\lambda))w_n
\end{aligned}$$

Substitution of these expressions into:

$$w_{n+1} = w_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

and collecting like powers of $h\lambda$ yields:

$$w_{n+1} = [1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4]w_n.$$

The amplification factor is therefore:

$$Q(h\lambda) = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4.$$

(b) The local truncation error is defined as

$$\tau_{n+1} = \frac{y(t_{n+1}) - z_{n+1}}{h}, \qquad (1)$$

where z_{n+1} is the numerical solution at t_{n+1}, obtained by starting from the exact value $y(t_n)$ in stead of w_n. Repeating the derivation under (a), with w_n replaced by $y(t_n)$, gives:

$$z_{n+1} = Q(h\lambda)y(t_n).$$

Using furthermore $y(t_{n+1}) = e^{h\lambda}y(t_n)$ in (1) it follows that

$$\tau_{n+1} = \frac{e^{h\lambda} - Q(h\lambda)}{h}y(t_n).$$

Canceling the first five terms of the expansion of $e^{h\lambda}$ against $Q(h\lambda)$, the required order of magnitude of τ_{n+1} follows.

(c) Use the transformation:

$$\begin{aligned} y_1 &= y \,, \\ y_2 &= y' \,, \end{aligned}$$

This implies that

$$\begin{aligned} y_1' &= y' = y_2 \,, \\ y_2' &= y'' = -qy_1 - py_2 + \sin t \,, \end{aligned}$$

So the matrix **A** and vector **g** are:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}; \quad \mathbf{g}(t) = \begin{pmatrix} 0 \\ \sin t \end{pmatrix}.$$

Characteristic equation: $\lambda^2 + p\lambda + q = 0$. $\lambda_{1,2} = \frac{-p \pm \sqrt{p^2-4q}}{2}$.

(d) Substitution of the values of p and q into the matrix **A** yields the eigenvalues $\lambda_{1,2} = -500 \pm i$. From the given drawing of the stability region the following can be inferred. Because the imaginary part is much smaller than the real part, an approximate stability condition can be obtained by simply neglecting the imaginary part. Then $h \leq 2.8/500 = 0.0056$ follows as the stability condition.

(e)

$$y'' + py' + qy = \sin t, \quad y(0) = y_0, \; y'(0) = y_0'. \tag{2}$$

After a short time the solution is close to a linear combination of $\sin t$ and $\cos t$, which is called a smooth solution.

The smooth solution can be integrated accurately by RK_4 with a 'large' step size: a step size of 0.1, let us say, would give an error of order 10^{-4} which is sufficient for most engineering purposes. However stability, governed by the eigenvalues, requires that the step size be restricted (see part (d)) to 0.0056. So the stability requirement forces us to choose a step size yielding an unnecessarily accurate solution, which is inefficient.

The Trapezoidal rule, on the other hand, is stable for all step sizes. So the step size is restricted by accuracy requirements only. The Trapezoidal rule has a global error of order h^2 such that a good accuracy may be expected for step sizes of about 0.01, which is much larger than the step size for RK4: 0.0056. An efficiency gain may be obtained in spite of the extra work connected with the implicitness of the method.

2. a Consider $y(x) = x^2$, then $y'(x) = 2x$ and $y''(x) = 2$, substitution into the differential equation yields

$$-y''(x) + xy = -2 + xx^2 = x^3 - 2, \tag{3}$$

hence $y(x) = x^2$ satisfies the differential equation. Next, we check the boundary conditions: $y'(0) = 2 \cdot 0 = 0$ and $y(1) = 1^2 = 1$ and hence also the boundary conditions are also satisfied. Hence, $y(x) = x^2$ is a solution of the boundary value problem.

b Using central differences for the second order derivative at a node $x_j = jh$, gives

$$y''(x_j) \approx \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} =: Q(h). \tag{4}$$

Here $y_j := y(x_j)$. Next, we will prove that this approximation is second order accurate, that is $|y''(x_j) - Q(h)| = O(h^2)$. Using Taylor's Theorem around $x = x_j$, gives

$$\begin{aligned} y_{j+1} &= y(x_j + h) = y(x_j) + hy'(x_j) + \tfrac{h^2}{2}y''(x_j) + \tfrac{h^3}{3!}y'''(x_j) + \tfrac{h^4}{4!}y''''(\eta_+), \\ y_{j-1} &= y(x_j - h) = y(x_j) - hy'(x_j) + \tfrac{h^2}{2}y''(x_j) - \tfrac{h^3}{3!}y'''(x_j) + \tfrac{h^4}{4!}y''''(\eta_-). \end{aligned} \tag{5}$$

Here, η_+ and η_- are numbers within the intervals (x_j, x_{j+1}) and (x_{j-1}, x_j), respectively. Substitution of these expressions into $Q(h)$ gives $|y''(x_j) - Q(h)| = O(h^2)$. Therewith, we obtain the following discretization formula for the internal grid nodes:

$$\frac{-w_{j-1} + 2w_j - w_{j+1}}{h^2} + x_j w_j = x_j^3 - 2. \tag{6}$$

Here w_j represents the numerical approximation of the solution y_j. To deal with the boundary $x = 0$, we use a virtual node at $x = -h$, and we define $y_{-1} := y(-h)$. Then, using central differences at $x = 0$ gives

$$0 = y'(0) \approx \frac{y_1 - y_{-1}}{2h} =: Q_b(h). \tag{7}$$

Using Taylor's Theorem, gives

$$\begin{aligned} &Q_b(h) = \\ &\frac{y(0) + hy'(0) + \frac{h^2}{2}y''(0) + \frac{h^3}{3!}y'''(\eta_+) - (y(0) - hy'(0) + \frac{h^2}{2}y''(0) - \frac{h^3}{3!}y'''(\eta_-))}{2h} = \\ &y'(0) + O(h^2). \end{aligned} \tag{8}$$

Again, we get an error of $O(h^2)$.

c With respect to the numerical approximation at the virtual node, we get

$$\frac{w_1 - w_{-1}}{2h} = 0 \Leftrightarrow w_{-1} = w_1. \tag{9}$$

The discretization at $x = 0$ is given by

$$\frac{-w_{-1} + 2w_0 - w_1}{h^2} = -2. \tag{10}$$

Substitution of equation (9) into the above equation, yields

$$\frac{2w_0 - 2w_1}{h^2} = -2. \tag{11}$$

Subsequently, we consider the boundary $x = 1$. To this extent, we consider its neighboring point x_{n-1}, here substitution of the boundary condition $w_n = y(1) = y_n = 1$ into equation (6), gives

$$\frac{-w_{n-2} + 2w_{n-1}}{h^2} + x_{n-1}w_{n-1} = x_{n-1}^3 - 2 + \frac{1}{h^2} = (1-h)^3 - 2 + \frac{1}{h^2}. \tag{12}$$

This concludes our discretization of the boundary conditions. In order to get a symmetric discretization matrix, one divides equation (11) by 2.

Next, we use $h = 1/3$, then, from equations (6, 11, 12), one obtains the following system

$$\begin{aligned} 9w_0 - 9w_1 &= -1 \\ -9w_0 + 18\tfrac{1}{3}w_1 - 9w_2 &= -\tfrac{53}{27} \\ -9w_1 + 18\tfrac{2}{3}w_2 &= \tfrac{197}{27}. \end{aligned} \tag{13}$$

d The truncation errors from the virtual grid point and internal points contain a third- and fourth order derivative, respectively (see part b). Since the exact solution is given by $y(x) = x^2$, the third and fourth order derivatives are zero. Hence, the error is zero. Therefore, the numerical solution is given by $w_0 = y_0 = 0$, $w_1 = y_1 = 1/9$ and $w_2 = y_2 = 4/9$.

Remark: This numerical solution can also be obtained from the solution of system (13).

e i Consider an interval of integration $[x_{j-1}, x_j]$, then the Rectangle Rule is given by

$$I_j^R = hf(x_{j-1}), \qquad h = x_j - x_{j-1}, \tag{14}$$

the Trapezoidal Rule is

$$I_j^T = \frac{h}{2}(f(x_{j-1}) + f(x_j)). \tag{15}$$

The composed integration Rules are derived by

$$I^{R,T} = h(I_1^{R,T} + I_2^{R,T} + \ldots + I_n^{R,T}) = \begin{cases} h(f(x_0) + \ldots + f(x_{n-1})), \\ h(\frac{f(x_0)}{2} + f(x_1) + \ldots + f(x_{n-1}) + \frac{f(x_n)}{2}), \end{cases} \tag{16}$$

for the Rectangle -and Trapezoidal Rule, respectively.

ii Using the Rectangle Rule, one obtains

$$I^R = \frac{1}{3} \cdot (0 + (\frac{1}{3})^2 + (\frac{2}{3})^2) = \frac{5}{27}. \tag{17}$$

From the Trapezoidal Rule, one gets

$$I^T = \frac{1}{3} \cdot (0 + (\frac{1}{3})^2 + (\frac{2}{3})^2 + \frac{1}{2}) = \frac{19}{54}. \tag{18}$$

f For a general number of subintervals, say n, the magnitude of the composed Rectangle- and Trapezoidal Rules, is bounded from above by

$$\begin{aligned} \varepsilon_R &\leq \frac{h}{2} \max_{x\in[0,1]} |y'(x)| \leq h = \frac{1}{n}, \\ \varepsilon_T &\leq \frac{h^2}{12} \max_{x\in[0,1]} |y''(x)| \leq \frac{h^2}{6} = \frac{1}{6n^2}. \end{aligned} \tag{19}$$

Here the exact solution $y(x) = x^2$ was used. Hence, the error from the Trapezoidal Rule is much smaller. Further, from the composed Rules, it is easy to see that the number of function evaluations for the composed Rectangle- and Trapezoidal Rules is, respectively, given by n and $n+1$. Since $\lim_{n\to\infty} \frac{n+1}{n} = 1$, it follows that the amount of work for the Trapezoidal Rule is not significantly higher than it is for the Rectangle Rule. Hence, it is more attractive to use the Trapezoidal Rule.