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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Tuesday March 31 2009, 14:00-17:00

1. (a) The local truncation error is given by

$$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h}. \tag{1}$$

Here we obtain y_{n+1} by a Taylor expansion around t_n:

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2}y''(t_n) + O(h^3). \tag{2}$$

For z_{n+1}, we obtain, after substitution of the predictor step for z^*_{n+1} into the corrector step

$$z_{n+1} = y_n + h\left((1-\theta)f(t_n, y_n) + \theta f(t_n + h, y_n + hf(t_n, y_n))\right) \tag{3}$$

After a Taylor expansion of $f(t_n+h, y_n+hf(t_n, y_n))$ around (t_n, y_n) one obtains:

$$z_{n+1} = y_n+h\left((1-\theta)f(t_n, y_n) + \theta(f(t_n, y_n) + h(\frac{\partial f(t_n, y_n)}{\partial t} + f(t_n, y_n)\frac{\partial f(t_n, y_n)}{\partial y})) + O(h^2)\right). \tag{4}$$

From the differential equation we know that:

$$y'(t_n) = f(t_n, y_n) \tag{5}$$

From the Chain Rule of Differentiation, we derive

$$y''(t_n) = \frac{df(t_n, y_n)}{dt} = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}y'(t_n) \tag{6}$$

after substitution of the differential equation one obtains:

$$y''(t_n) = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}f(t_n, y_n) \tag{7}$$

This implies that $z_{n+1} = y_n + hy'(t_n) + \theta h^2 y''(t_n)$. Subsequently, it follows that

$$y_{n+1} - z_{n+1} = O(h^2), \text{ and, hence } \tau_{n+1}(h) = \frac{O(h^2)}{h} = O(h) \text{ for } 0 \le \theta \le 1, \tag{8}$$

$$y_{n+1} - z_{n+1} = O(h^3), \text{ and, hence } \tau_{n+1}(h) = \frac{O(h^3)}{h} = O(h^2) \text{ for } \theta = \frac{1}{2}. \tag{9}$$

(b) Consider the test equation $y' = \lambda y$, then, herewith, one obtains

$$\begin{aligned} w_{n+1}^* &= w_n + h\lambda w_n = (1+h\lambda)w_n, \\ w_{n+1} &= w_n + h((1-\theta)\lambda w_n + \theta\lambda w_{n+1}^*) = \\ &= w_n + h((1-\theta)\lambda w_n + \theta\lambda(w_n + h\lambda w_n)) = (1+h\lambda+\theta(h\lambda)^2)w_n. \end{aligned} \tag{10}$$

Hence the amplification factor is given by

$$Q(h\lambda) = 1 + h\lambda + \theta(h\lambda)^2. \tag{11}$$

(c) We start this exercise by using the following vector:

$$x_1 = y$$

$$x_2 = y'$$

From this it follows that

$$x_1' = y' = x_2$$
$$x_2' = y'' = -2y' - 2y + t = -2x_2 - 2x_1 + t$$

where we have used the second order differential equation. We can write this as follows in matrix-vector notation:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix}$$

So it follows that $A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$ and $f(t) = 0$ and $g(t) = t$.

(d) In order to do one step we first note that

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The predictor step with $h = 1$ now gives:

$$w_1^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1\left(\begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Finally the correction step with $\theta = \frac{1}{2}$ leads to

$$w_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \frac{1}{2}\left(\begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

(e) Compute the eigenvalues of matrix $\begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$. To do this we compute the determinant of $\begin{pmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{pmatrix}$, which is equal to $\lambda^2 + 2\lambda + 2$. The roots of this polynomial are equal to $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$. Since $\lambda_2 = \bar{\lambda_1}$ it is sufficient to consider λ_1 only. For $h = 2$ we obtain $h\lambda_1 = -2 + 2i$. This implies that

$$Q(h\lambda_1) = 1 + h\lambda_1 + \theta(h\lambda_1)^2$$
$$Q(h\lambda_1) = 1 + (-2 + 2i) + \theta(-2 + 2i)^2$$
$$Q(h\lambda_1) = 1 - 2 + 2i + \theta(4 - 8i - 4) = -1 + i(2 - 8\theta)$$

In order to check that $|Q(h\lambda_1)| \leq 1$, we compute the modulus of $Q(h\lambda_1)$, which is equal to

$$\sqrt{1^2 + (2 - 8\theta)^2}$$

It is easy to see that this is only less than or equal to 1 if $\theta = \frac{1}{4}$.

2. (a) First, we do one fixed point iteration:

$$p_1 = g_1(p_0) = \frac{\pi\sqrt{2}}{8\sin(\frac{\pi}{2})} = \frac{\pi\sqrt{2}}{8},$$

$$q_1 = g_2(q_0) = \frac{\pi}{2} - (\frac{\pi}{2}\sin(\frac{\pi}{2}) - \frac{\pi\sqrt{2}}{8}) = \frac{\pi\sqrt{2}}{8}.$$

Let $\tilde{x}$ be a fixed point of $g_1(x)$, then

$$\tilde{x} = \frac{\pi\sqrt{2}}{8\sin(\tilde{x})}.$$

Multiplication by $\sin(\tilde{x})$, gives

$$\tilde{x}\sin(\tilde{x}) = \frac{\pi\sqrt{2}}{8},$$

hence $\tilde{x}$ is a solution of the original problem. Subsequently, we assume $\tilde{x}$ to be a fixed point of $g_2(x)$, then

$$\tilde{x} = \tilde{x} - (\tilde{x}\sin(\tilde{x}) - \frac{\pi\sqrt{2}}{8}).$$

Subtraction of $\tilde{x}$ in the above equation, gives

$$\tilde{x}\sin(\tilde{x}) - \frac{\pi\sqrt{2}}{8} = 0,$$

hence $\tilde{x}$ is a solution of the original problem. □

(b) From convergence follows that $\lim_{k\to\infty} p_k = p = \dfrac{\pi}{4}$. Since ξ_k has a value between p and p_k, we obtain using continuity of $g'(x)$ (and from the Squeeze Theorem that $\xi_k \to p = \frac{\pi}{4}$ as $k \to \infty$), that $\lim_{k\to\infty} g'(\xi_k) = g'(p) = g'(\dfrac{\pi}{4})$. Herewith, we obtain after differentiation

$$\lim_{k\to\infty} |g_1'(\xi)| = |g_1'(\frac{\pi}{4})| = \left| -\frac{\pi\sqrt{2}}{8}\frac{cos(\frac{\pi}{4})}{sin^2(\frac{\pi}{4})} \right| = \frac{\pi}{4} \approx 0.7854,$$

$$\lim_{k\to\infty} |g_2'(\xi)| = |g_2'(\frac{\pi}{4})| = |1 - sin(\frac{\pi}{4}) - \frac{\pi}{4}cos(\frac{\pi}{4})| = |1 - \frac{4+\pi}{4\sqrt{2}}| \approx 0.2627.$$

Hence, $|g_2'(p)| < |g_1'(p)|$, Using $|p - p_{k+1}| = |g'(\xi_k)||p - p_k|$ gives $|p - p_{k+1}| = |g'(p)||p - p_k|$ as $k \to \infty$. Hence, using the function $g_2(x)$ for the fixed point method, gives a faster convergence. □

(c) We search a zero of the function

$$f(x) = x\sin(x) - \frac{\pi\sqrt{2}}{8}.$$

The derivative is given by

$$f'(x) = \sin(x) + x\cos(x).$$

Herewith, we get

$$z_1 = \frac{\pi}{2} - \frac{\frac{\pi}{2}\sin(\frac{\pi}{2}) - \frac{\pi\sqrt{2}}{8}}{\sin(\frac{\pi}{2}) + \frac{\pi}{2}\cos(\frac{\pi}{2})} = \frac{\pi\sqrt{2}}{8} \approx 0.554.$$

□

(d) The Newton-Raphson iteration method can be derived using a graph of a function, in which the zero of the tangent at z_k on $f(x)$ defines z_{k+1}. We consider a linearization of $f(x)$ around z_k:

$$L(x) := f(z_k) + (x - z_k)f'(z_k),$$

and determine its zero, that is $L(z_{k+1}) = 0$, this gives

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}, \text{ provided that } f'(z_k) \neq 0,$$

□

(e) We consider a Taylor polynomial around z_k, to express z

$$0 = f(z) = f(z_k) + (z - z_k)f'(z_k) + \frac{(z - z_k)^2}{2}f''(\xi_k), \tag{12}$$

for some ξ_k between z and z_k. Note that this form gives the exact representation. Subsequently, we consider the Newton-Raphson approximation

$$0 = L(z_{k+1}) = f(z_k) + (z_{k+1} - z_k)f'(z_k). \tag{13}$$

Subtraction of these two above equations gives

$$z_{k+1} - z = \frac{(z_k - z)^2}{2} \frac{f''(\xi_k)}{f'(z_k)}, \text{ provided that } f'(z_k) \neq 0, \tag{14}$$

and hence

$$|z_{k+1} - z| = \frac{(z_k - z)^2}{2} \left|\frac{f''(\xi_k)}{f'(z_k)}\right|, \text{ provided that } f'(z_k) \neq 0, \tag{15}$$

Using $z_k \to z$, $\xi_k \to z$ as $k \to \infty$ and continuity of $f(x)$ up to at least the second derivative, we arrive at $K = |\frac{f''(z)}{f'(z)}| = \frac{8-\pi}{4+\pi} \approx 0.6803$. □