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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Tuesday January 27 2009, 9:00-12:00

1. a) The local truncation error is defined as

$$\tau_{n+1} = \frac{y_{n+1} - z_{n+1}}{h}, \tag{1}$$

where y_{n+1} is the exact solution at t_{n+1} and z_{n+1} the value obtained by applying the given method at the exact solution point (t_n, y_n):

$$\begin{aligned} k_1 &= hf(t_n, y_n) \\ k_2 &= hf(t_n + h, y_n + k_1) \\ z_{n+1} &= y_n + \beta\ k_1 + (1 - \beta)\ k_2. \end{aligned} \tag{2}$$

Both y_{n+1} and z_{n+1} have to be expanded into a Taylor series at the point (t_n, y_n). To start with z_{n+1}, k_1 and k_2 are substituted into the corrector part (2):

$$z_{n+1} = y_n + \beta\ hf(t_n, y_n) + (1 - \beta)\ hf(t_n + h, y_n + hf(t_n, y_n)). \tag{3}$$

Next $f(t_n + h, y_n + hf(t_n, y_n))$ is expanded:

$$\begin{aligned} f(t_n + h, y_n + hf(t_n, y_n)) &= f(t_n, y_n) + h\frac{\partial f}{\partial t}(t_n, y_n) + hf(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n) + \dots \\ &= y'_n + h[\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}](t_n, y_n) + O(h^2), \end{aligned} \tag{4}$$

using the differential equation $y' = f(t, y)$.
In this expression $[\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial t}](t_n, y_n)$ can be replaced by $y''(t_n) = y''_n$, for

$$y'' = \frac{dy'}{dt} = \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y},$$

again using $y' = f(t, y)$ in the last step.
As a result, (4) becomes:

$$f(t_n + h, y_n + hf(t_n, y_n)) = y'_n + hy''_n + O(h^2).$$

Substitution of this expression into (3) gives:

$$\begin{aligned} z_{n+1} &= y_n + \beta\ hy'_n + (1 - \beta)\ h\ (y'_n + hy''_n + O(h^2)) \\ &= y_n + hy'_n + (1 - \beta)h^2 y''_n + O(h^3). \end{aligned}$$

Substitution of this expansion, together with the expansion for y_{n+1}:

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2 y''_n + O(h^3),$$

into (1) yields:

$$\begin{aligned}\tau_{n+1} &= \frac{y_n + hy'_n + \frac{1}{2}h^2 y''_n + O(h^3) - [y_n + hy'_n + (1-\beta)h^2 y''_n + O(h^3)]}{h} \\ &= (\beta - \frac{1}{2})\ h\ y''_n + O(h^2)\end{aligned}$$

It turns out that the truncation error is O(h), except for $\beta = \frac{1}{2}$. Note that the predictor-corrector method is just Modified Euler for $\beta = \frac{1}{2}$.

b) The amplification factor is found by applying the method to the homogeneous test equation $y' = \lambda y$:

$$\begin{aligned}k_1 &= h\lambda w_n \\ k_2 &= h\lambda(w_n + h\lambda w_n) = h\lambda(1 + h\lambda)w_n \\ w_{n+1} &= w_n + \beta\ h\lambda w_n + (1-\beta)\ h\lambda(1 + h\lambda)w_n \\ &= [1 + h\lambda + (1-\beta)(h\lambda)^2]w_n.\end{aligned}$$

The amplification factor $Q(h\lambda)$ is seen to be $1 + h\lambda + (1-\beta)(h\lambda)^2$.

c) To derive the stability condition we need the eigenvalues of the system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}.$$

These are purely imaginary: $\lambda_{1,2} = \pm i$, as can be seen easily.
For stability we require $|Q(\pm hi)| < 1$ or, more conveniently,
$|Q(\pm hi)|^2 < 1$.
From c):

$$\begin{aligned}|1 \pm hi + (1-\beta)(\pm hi)^2|^2 &< 1 \leftrightarrow \\ |1 - (1-\beta)h^2 \pm hi|^2 &< 1 \leftrightarrow \\ (1 - (1-\beta)h^2)^2 + h^2 &< 1 \leftrightarrow \\ 1 - 2(1-\beta)h^2 + (1-\beta)^2 h^4 + h^2 &< 1 \leftrightarrow \\ (1-\beta)^2 h^2 &< 2(1-\beta) - 1 = 1 - 2\beta.\end{aligned}$$

(Note: the squared modulus of a complex number equals the sum of the squares of it's real and imaginary part.)
It now follows that

$$h^2 < \frac{1 - 2\beta}{(1-\beta)^2}$$

is required for stability.
Clearly, stability is possible only for $\beta < \frac{1}{2}$.

d) We have optimal stability if the upper bound for h is as large as possible. So we have to investigate the behavior of the function $g(\beta) = \frac{1-2\beta}{(1-\beta)^2}$ for $\beta < \frac{1}{2}$. The derivative of $g(\beta)$ is given: $\frac{-2\beta}{(1-\beta)^2}$. This derivative is positive for $\beta < 0$ and negative for $0 < \beta < \frac{1}{2}$. So $g(\beta)$ assumes its maximum for $\beta = 0$, $g(0)$ being 1. The optimal stability condition for the considered system is therefore $h < 1$.

e) The stability bound of the optimal predictor-corrector method is $h < 1$, as found under d. The stability bound of Runge-Kutta is $h < 2\sqrt{2} \approx 2.8$, as can be read off from the included stability region. Since the optimal predictor-corrector method uses 2 function evaluations per time step it appears that per function evaluation a distance of 0.5 can be covered. The Runge-Kutta method uses 4 function evaluations per time step so per function evaluation a distance of 0.7 can be covered. Since accuracy is not an issue it is more efficient to use the Runge-Kutta method.

2. a After discretization by the use of finite differences one obtains

$$\frac{-w_{i-1} + 2w_i - w_{i+1}}{h^2} + x_i w_i = x_i^2. \tag{5}$$

The truncation error is defined by

$$e_i = \frac{-y_{i-1} + 2y_i - y_{i+1}}{h^2} + x_i y_i - x_i^2. \tag{6}$$

Taylor series of y_{i-1} and y_{i+1} around x_i, gives

$$\begin{aligned} y_{i+1} &= y_i + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y'''(x_i) + \frac{h^4}{4!}y''''(x_i) + O(h^5), \\ y_{i-1} &= y_i - hy'(x_i) + \frac{h^2}{2!}y''(x_i) - \frac{h^3}{3!}y'''(x_i) + \frac{h^4}{4!}y''''(x_i) - O(h^5), \end{aligned} \tag{7}$$

Substitution of the above expressions into the definition of the truncation error gives

$$\varepsilon_i = -y''(x_i) + O(h^2) + x_i y(x_i) - x_i^2. \tag{8}$$

Using the differential equation $-y'' + xy = x^2$ finally gives

$$\varepsilon_i = O(h^2). \tag{9}$$

b For this case we have $h = 0.25$, for the points $j \in \{1, 2, 3\}$, the discretization with $w_0 = 1$ and $w_4 = 0$:

$$\begin{aligned} &32w_1 - 16w_2 + \frac{1}{4}w_1 = \frac{1}{16} + 16, \\ &-16w_1 + 32w_2 - 16w_3 + \frac{1}{2}w_2 = \frac{1}{4}, \\ &-16w_2 + 32w_3 + \frac{3}{4}w_3 = \frac{9}{16}. \end{aligned} \tag{10}$$

Hence in matrix-vector form:

$$\begin{pmatrix} 32.25 & -16 & 0 \\ -16 & 32.5 & -16 \\ 0 & -16 & 32.75 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 16.0625 \\ 0.25 \\ 0.5625 \end{pmatrix} \tag{11}$$

c Since $h = \frac{1}{3}$, we have $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$ and $x_3 = 1$. Using linear interpolation, two adjacent gridpoints are taken into account. The minimum error is attained when the gridpoints x_1 and x_2 are used. The linear interpolation formula using points x_1 and x_2, gives:

$$p(x) = \frac{x - x_2}{x_1 - x_2} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2) \ .$$

$$P(0.4) = \frac{0.4 - \frac{2}{3}}{\frac{1}{3} - \frac{2}{3}} \cdot 0.4444 + \frac{0.4 - \frac{1}{3}}{\frac{2}{3} - \frac{1}{3}} \cdot 0.7778 = 0.5111 \tag{12}$$

The magnitude of the local truncation error is given by

$$\left| \frac{(x - x_1)(x - x_2)}{2} y''(\xi) \right| = \left| \frac{(0.4 - 1/3)(0.4 - 2/3)}{2} \cdot 1 \right| = 0.0089. \tag{13}$$

d The magnitude of the truncation error is given by

$$\left| \frac{y_2 - y_1}{h} - y'(x_2) \right| = \left| \frac{y(x_2) - y(x_2) + hy'(x_2) - \frac{h^2}{2} y''(\xi)}{h} - y'(x_2) \right| = \frac{h}{2} \left| y''(\xi) \right| = \frac{h}{2} = \frac{1}{6}. \tag{14}$$

e The additional error is given by

$$\left| \frac{y_2 - y_1}{h} - \frac{w_2 - w_1}{h} \right| \leq \frac{2\varepsilon}{h} = \frac{2 \cdot 0.01}{\frac{1}{3}} = 0.06. \tag{15}$$