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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Friday August 28 2009, 14:00-17:00

1. (a) The local truncation error is given by

$$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h}, \tag{1}$$

in which we determine y_{n+1} by the use of Taylor expansions around t_n:

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2}y''(t_n) + O(h^3). \tag{2}$$

We bear in mind that

$$\begin{aligned} y'(t_n) &= f(t_n, y_n) \\ y''(t_n) &= \frac{df(t_n,y_n)}{dt} = \frac{\partial f(t_n,y_n)}{\partial t} + \frac{\partial f(t_n,y_n)}{\partial y}y'(t_n) = \\ &\frac{\partial f(t_n,y_n)}{\partial t} + \frac{\partial f(t_n,y_n)}{\partial y}f(t_n,y_n). \end{aligned} \tag{3}$$

Hence

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2}\left(\frac{\partial f(t_n,y_n)}{\partial t} + \frac{\partial f(t_n,y_n)}{\partial y}f(t_n,y_n)\right) + O(h^3). \tag{4}$$

After substitution of the predictor $z^*_{n+1} = y_n + hf(t_n, y_n)$ into the corrector, and after using a Taylor expansion around (t_n, y_n), we obtain for z_{n+1}

$$\begin{aligned} &z_{n+1} = y_n + \tfrac{h}{2}\left(f(t_n,y_n) + f(t_n + h, y_n + hf(t_n,y_n))\right) = \\ &y_n + \frac{h}{2}\left(f(t_n,y_n) + f(t_n,y_n) + h\left(\frac{\partial f(t_n,y_n)}{\partial t} + f(t_n,y_n)\frac{\partial f(t_n,y_n)}{\partial y}\right) + O(h^2)\right). \end{aligned} \tag{5}$$

Herewith, one obtains

$$y_{n+1} - z_{n+1} = O(h^3), \text{ and hence } \tau_{n+1}(h) = \frac{O(h^3)}{h} = O(h^2). \tag{6}$$

(b) Let $x_1 = y$ and $x_2 = y'$, then $y'' = x_2'$, and hence

$$\begin{aligned} x_2' + x_1 &= \sin(t), \\ x_2 &= x_1'. \end{aligned} \tag{7}$$

We write this as

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -x_1 + \sin(t). \end{aligned} \tag{8}$$

Finally, this is represented in the following matrix-vector form:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(t) \end{pmatrix}. \tag{9}$$

In which, we have the following matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 \\ \sin(t) \end{pmatrix}$. The initial conditions are defined by $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(c) Application of the Modified Euler method to the system $\underline{x}' = A\underline{x} + \underline{f}$, gives

$$\begin{aligned} \underline{w}_1^* &= \underline{w}_0 + h\left(A\underline{w}_0 + \underline{f}_0\right), \\ \underline{w}_1 &= \underline{w}_0 + \tfrac{h}{2}\left(A\underline{w}_0 + f_0 + A\underline{w}_1^* + \underline{f}_1\right). \end{aligned} \tag{10}$$

With the initial condition $\underline{w}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $h = 0.1$, this gives the following result for the predictor

$$\underline{w}_1^* = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{10}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 6/5 \\ 19/10 \end{pmatrix}. \tag{11}$$

The corrector is calculated as follows

$$\begin{aligned} \underline{w}_1 &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \tfrac{1}{20}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 6/5 \\ 19/10 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(\frac{1}{10}) \end{pmatrix}\right) = \\ &= \begin{pmatrix} 1.19500 \\ 1.89492 \end{pmatrix} \end{aligned} \tag{12}$$

(d) Consider the test equation $y' = \lambda y$, then one gets

$$\begin{aligned} w_{n+1}^* &= w_n + h\lambda w_n = (1 + h\lambda)w_n, \\ w_{n+1} &= w_n + \frac{h}{2}(\lambda w_n + \lambda w_{n+1}^*) = \\ &= w_n + \frac{h}{2}(\lambda w_n + \lambda(w_n + h\lambda w_n)) = (1 + h\lambda + \frac{(h\lambda)^2}{2})w_n. \end{aligned} \tag{13}$$

Hence the amplification factor is given by

$$Q(h\lambda) = 1 + h\lambda + \frac{(h\lambda)^2}{2}. \tag{14}$$

(e) To this extent, we determine the eigenvalues of the matrix A. Subsequently, the eigenvalues are substituted into the amplification factor. The eigenvalues of the matrix A are given by $\lambda_1 = i$ and $\lambda_2 = -i$. Since both eigenvalues lead to the same modulus of the amplification factor we only consider the first eigenvalue. Substitution of this eigenvalues into the amplification factor gives

$$Q(hi) = 1 + hi + \frac{1}{2}h^2 i^2 = (1 - \frac{1}{2}h^2) + hi. \tag{15}$$

Then, the square of the modulus of the amplification factor is given by

$$|Q(hi)|^2 = (1 - \frac{1}{2}h^2)^2 + h^2 = 1 + \frac{1}{4}h^4 > 1, \quad \text{for all } h > 0. \tag{16}$$

From the above observation, we immediately see that the amplification factor is always larger than one, hence the Modified Euler method applied to our currently studied system is never stable.

Remark: This could also be concluded immediately from the stability region. Since the eigenvalues are located on the imaginary axis, and since the Modified Euler method is always unstable if an eigenvalue is located on the imaginary axis (not including the origin).

2. (a) The formula for $L_1(x)$ and $L_2(x)$ are:

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \text{ and } L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

(b) For the approximation we note that $x_0 = 0$, $x_1 = 0.1$ and $x_2 = 0.4$. After substitution in the formula one obtains:

$$L_0(0.2) = \frac{(0.1)(-0.2)}{(-0.1)(-0.4)} = \frac{-2}{4} = -\frac{1}{2}$$

$$L_1(0.2) = \frac{(0.2)(-0.2)}{(0.1)(-0.3)} = \frac{-4}{3} = 1\frac{1}{3}$$

$$L_2(0.2) = \frac{(0.2)(0.1)}{(0.4)(0.3)} = \frac{2}{12} = \frac{1}{6}$$

The approximation is given by: $-\frac{1}{2} \cdot 0 + 1\frac{1}{3} \cdot 0.0953 + \frac{1}{6} \cdot 0.3365 = 0.1832$.

(c) From the definition of the second order Lagrange interpolation formula it appears that $|\hat{p}(x) - p(x)| < max_{x \in [x_0, x_2]} |L_2(x)| \epsilon$. If we can prove that $max_{x \in [x_0, x_2]} |L_2(x)| \leq 1$ then the inequality is correct. We first substitute the data:

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x)(x - 0.1)}{(0.4)(0.3)}$$

To determine $max_{x\in[0,0.4]}|L_2(x)|$ we consider $max_{x\in[0,0.1]}\frac{(x)(0.1-x)}{0.12} \leq \frac{0.05\cdot 0.05}{0.12} = 0.0208$ and $max_{x\in[0.1,0.4]}\frac{(x)(x-0.1)}{0.12}$. This function does not have an internal maximum, so $max_{x\in[0.1,0.4]}\frac{(x)(x-0.1)}{0.12} \leq \frac{0.4\cdot 0.3}{0.12} = 1$, which implies that $max_{x\in[x_0,x_2]}|L_2(x)| \leq 1$.

(d) From the Taylor polynomial it follows that $f(h) = f(0) + hf'(0) + O(h^2)$. This implies $f'(0) - \frac{f(h)-f(0)}{h} = f'(0) - \frac{f(0)+hf'(0)+O(h^2)-f(0)}{h} = O(h)$.

(e) After the Taylor polynomial and error term is made for $f(0)$, $f(h)$ and $f(2h)$ we come to the following result:

$$\begin{aligned} f(0) &= f(0) && (17)\\ f(h) &= f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + O(h^3) && (18)\\ f(2h) &= f(0) + 2hf'(0) + \frac{4h^2}{2!}f''(0) + O(h^3) && (19) \end{aligned}$$

Using formula $\alpha_0 f(0) + \alpha_1 f(h) + \alpha_2 f(2h)$ we try to find the values of α_0, α_1 and α_2 such that $f'(0) - (\alpha_0 f(0) + \alpha_1 f(h) + \alpha_2 f(2h)) = O(h^2)$. Using the above given Taylor polynomials we obtain the following system:

$$\begin{aligned} f(0):&\quad \alpha_0 + \alpha_1 + \alpha_2 = 0\\ f'(0):&\quad \alpha_1 h + \alpha_2 2h = 1\\ f''(0):&\quad \alpha_1\frac{h^2}{2!} + \alpha_2 2h^2 = 0 \end{aligned}$$

After solution the following values are obtained: $\alpha_0 = \frac{-3}{2h}$, $\alpha_1 = \frac{4}{2h}$ and $\alpha_2 = \frac{-1}{2h}$. The formula is

$$\frac{-3f(0) + 4f(h) - f(2h)}{2h}.$$

(f) The exact derivative is $f'(x) = -\sin(x)+1$, so $f'(0) = 1$. The Forward difference approximation leads to

$$\frac{f(h) - f(0)}{h}\frac{1.0950 - 1}{0.1} = 0.95,$$

where the error is 1-0.95 = 0.05. Using the second order formula one obtains:

$$\frac{-3f(0) + 4f(h) - f(2h)}{2h} = \frac{-3\cdot 1 + 4\cdot 1.0950 - 1.1801}{0.2} = 0.9995,$$

where the error is 1-0.9995 = 0.0005. We prefer the second order formula because the error is much less than the error for the Forward difference approximation.

(g) The rounding error is $\frac{h}{2!}f''(\xi)$ with $\xi \in [0, 0.1]$. In this example is $f''(x) = -\cos(x)$ so the absolute value of the truncation error is less than $\frac{0.1}{2}cos(0) = 0.5\cdot 10^{-1}$. For the absolute rounding error it appears that this is less than $\frac{|\hat{f}(0.1)-f(0.1)|}{h} = 0.5\cdot 10^{-3}$. Note that the truncation error is larger than the rounding error, so it makes sense to reduce the step size.