DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)

Tuesday January 29 2008, 9:00-12:00

1.

a) In the general formulation,

$$w_{n+1} = w_n + hf(t_{n+1}, w_{n+1}),$$

$f(t_{n+1}, w_{n+1})$ is replaced by λw_{n+1}:

$$w_{n+1} = w_n + h\lambda w_{n+1}.$$

Solving for w_{n+1}, we find

$$w_{n+1} = \frac{1}{1-h\lambda} w_n.$$

It follows, by definition, that the amplification factor equals

$$Q(h\lambda) = \frac{1}{1-h\lambda}.$$

b) For a λ with a negative real part, we have

$$|Q(h\lambda)| = |\frac{1}{1-hRe(\lambda) - ihIm(\lambda)}|$$

$$= \frac{1}{\sqrt{(1-hRe(\lambda))^2 + (hIm(\lambda))^2}}$$

Since $Re(\lambda)) \leq 0$ it appears that $1 - hRe(\lambda) \geq 1$, so

$$|Q(h\lambda)| = \frac{1}{\sqrt{(1-hRe(\lambda))^2 + (hIm(\lambda))^2}} \leq 1$$

independent of h. Hence, the BE method is unconditionally stable.

c) The local truncation error is defined as

$$\tau_{n+1} = \frac{y_{n+1} - \bar{w}_{n+1}}{h}. \quad (1)$$

The exact solution of the test equation $y' = \lambda y$ can be written as $y_n e^{\lambda(t-t_n)}$; hence at $t = t_n + h$ we have $y_{n+1} = e^{h\lambda} y_n$.

The quantity $\bar{w}_{n+1}$ is defined as the numerical solution on the interval (t_n, t_{n+1}), starting from the exact value y_n. So, for the test equation :

$$\bar{w}_{n+1} = y_n + h\lambda\bar{w}_{n+1},$$

such that (compare (a)) $\bar{w}_{n+1} = \frac{1}{1-h\lambda}y_n$. Now insert both expressions into the definition (1):

$$\tau_{n+1} = \frac{e^{h\lambda} - \frac{1}{1-h\lambda}}{h}y_n,$$

and replace $e^{h\lambda}$ and $\frac{1}{1-h\lambda}$ by their expansions $1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \ldots$, respectively $1+h\lambda+h^2\lambda^2+\ldots$. The first two terms of the expansions cancel and we are left with

$$\tau_{n+1} = \frac{\frac{1}{2}h^2\lambda^2 + \ldots - (h^2\lambda^2 + \ldots)}{h} = O(h),$$

which proves that BE is $O(h)$.

d) Calling $x_1 = y$ and $x_2 = y'$, the first differential equation follows directly: $x_1' = x_2$. Note that $x_2' = y''$, substituting $y'' = -1000.5y' - 500y = -1000.5x_2 - 500x_1$ yields the second equation: $x_2' = -500x_1 - 1000.5x_2$. Using matrix-vector notation, the two equations are compiled as $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} 0 & 1 \\ -500 & -1000.5 \end{bmatrix}.$$

To find the eigenvalues, the determinant of

$$\begin{bmatrix} -\lambda & 1 \\ -500 & -1000.5 - \lambda \end{bmatrix},$$

is put equal to 0. This leads to the quadratic equation $\lambda^2 + 1000.5\lambda + 500 = 0$ with roots $\lambda_1 = -1000$ and $\lambda_2 = -0.5$.
Note that both eigenvalues are negative and that $|\lambda_1|$ is much larger than $|\lambda_2|$.

e) For an $O(h^p)$- method the error at time t is estimated by the general formula ((6.54) of the lecture notes)

$$y(t) - w(t, \frac{h}{2}) \approx \frac{w(t, \frac{h}{2}) - w(t, h)}{2^p - 1}.$$

Using $p = 1$ and $h = 0.06$, application of this formula to the given table values yields

$$\begin{aligned} y_2 - w_2(3.6, 0.03) &\approx w_2(3.6, 0.03) - w_2(3.6, 0.06) \\ &= -0.167694 + 0.169903 \approx 0.002209 \end{aligned}$$

as the BE-error for the second component (the derivative of the solution) at $t = 3.6$. This is well within the given tolerance of 0.0025.

f) The stability condition of Forward Euler, applied to $y' = \lambda y$, reads

$$h < \frac{2}{|\lambda|}.$$

To apply this condition to the system derived in (d), we have to substitute its (in absolute value largest) eigenvalue -1000 for λ. It follows that h has to satisfy the condition $h < 0.002$.

g) For a step size which is close to its maximal value 0.002, Forward Euler produces a result with a given error of 0.000112, far less than the required accuracy of 0.0025. From the point of view of efficiency we would like to increase the step size but that is impossible because of stability requirements. Because both Euler methods are $O(h)$ their accuracy is comparable, but Backward Euler is unconditionally stable and hence, the step size can be increased at will. The error estimate in (e) has shown that a step size of 0.03, at least 15 times larger than the maximal stable step size of Euler Forward, is sufficient to meet the required accuracy. So, Euler Backward is the most suitable method.

2. (a) A fixed point p satisfies the equation $p = g(p)$. Substitution gives: $p = p + \frac{1}{2} - \frac{1}{2}p^2$. Rewriting this expression gives:

$$\begin{aligned} 0 &= \frac{1}{2} - \frac{1}{2}p^2 \\ \frac{1}{2}p^2 &= \frac{1}{2} \\ p^2 &= 1 \\ p &= \pm 1. \end{aligned}$$

On the other hand $f(p) = 0$ gives

$$\begin{aligned} \frac{p^2}{1+p^2} - \frac{1}{2} &= 0 \\ \frac{p^2}{1+p^2} &= \frac{1}{2} \\ 2p^2 &= 1 + p^2 \\ p^2 &= 1 \\ p &= \pm 1. \end{aligned}$$

So both expressions leads to the same solutions.

The fixed point iteration is defined by: $p_{i+1} = g(p_i)$. Starting with $p_0 = \frac{1}{2}$ one obtains:

$$\begin{aligned} p_1 &= 0.875, \\ p_2 &= 0.9922, \\ p_3 &= 1. \end{aligned}$$

(b) For the convergence two conditions should be satisfied:

- $g(p) \in [\frac{1}{2}, 1]$ for all $p \in [\frac{1}{2}, 1]$.
- $|g'(p)| \leq k < 1$ for all $p \in [\frac{1}{2}, 1]$.

Since $g(p) = p + \frac{1}{2} - \frac{1}{2}p^2$, the derivative is $g'(p) = 1 - p$. Note that $g'(p) \geq 0$ for all $p \in [\frac{1}{2}, 1]$. This implies that

$$0.875 = g(\frac{1}{2}) \leq g(p) \leq g(1) = 1,$$

so the first condition holds. For the second condition we note that $|g'(p)| = |1 - p| \leq \frac{1}{2} = k < 1$ for all $p \in [\frac{1}{2}, 1]$, so the second conditions is also satisfied, which implies that the fixed point iteration is convergent for all $p_0 \in [\frac{1}{2}, 1]$.

(c) Since $g'(p) = 1 - p$ it follows that

$$|g'(-1)| = |2| = 2 > 1,$$

so the method is divergent.

(d) Graphically the Newton-Raphson method is given in Figure 1. The tangent in

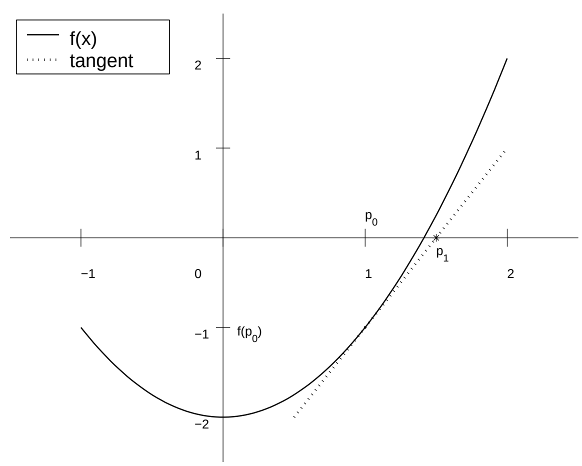

Figure 1: The Newton-Raphson method

$(p_0, f(p_0))$ is given by:

$$l(x) = f(p_0) + (x - p_0)f'(p_0).$$

Taking $l(p_1) = 0$ leads to

$$f(p_0) + (p_1 - p_0)f'(p_0) = 0.$$

Rewriting gives $p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}$.

(e) Starting with $p_0 = \frac{1}{2}$ we note that

$$\begin{aligned} f(p) &= \frac{p^2}{1+p^2} - \frac{1}{2}, \\ f'(p) &= \frac{2p}{(1+p^2)^2}. \end{aligned}$$

Substituting this into the formula gives

$$p_1 = \frac{1}{2} - \frac{(\frac{\frac{1}{4}}{1+\frac{1}{4}} - \frac{1}{2})}{\frac{1}{(1+\frac{1}{4})^2}} = 0.96875$$

(f) Note that

$$|\hat{p}_{i+1} - p_{i+1}| = |\hat{p}_i - \frac{\hat{f}(\hat{p}_i)}{\hat{f}'(\hat{p}_i)} - (p_i - \frac{f(p_i)}{f'(p_i)})|.$$

From the assumptions $\hat{p}_i = p_i$ and $\hat{f}'(p_i) = f'(p_i)$ it follows that

$$|\hat{p}_{i+1} - p_{i+1}| \leq \frac{|\hat{f}(p_i) - f(p_i)|}{|f'(p_i)|} \leq \frac{\epsilon}{|f'(p_i)|} \leq 4\epsilon,$$

since

$$|f'(p)| = \frac{2p}{(1+p^2)^2} \geq \frac{2 \cdot \frac{1}{2}}{(1+1)^2} = \frac{1}{4}.$$