DELFT UNIVERSITY OF TECHNOLOGY
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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)

Friday August 29 2008, 14:00-17:00

1. (a) The amplification factor can be derived as follows. Consider the test equation $y' = \lambda y$. Application of the trapezoidal rule to this equation gives:

$$w_{j+1} = w_j + \frac{h}{2}\left(\lambda w_j + \lambda w_{j+1}\right) \tag{1}$$

Rearranging of w_{j+1} and w_j in (1) yields

$$\left(1 - \frac{h}{2}\lambda\right) w_{j+1} = \left(1 + \frac{h}{2}\lambda\right) w_j.$$

It now follows that

$$w_{j+1} = \frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda} w_j,$$

and thus

$$Q(h\lambda) = \frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda}.$$

(b) The definition of the local truncation error is

$$\tau_{j+1} = \frac{y_{j+1} - Q(h\lambda)y_j}{h}.$$

The exact solution of the test equation is given by

$$y_{j+1} = e^{h\lambda} y_j.$$

Combination of these results shows that the local truncation error of the test equation is determined by the difference between the exponential function and the amplification factor $Q(h\lambda)$

$$\tau_{j+1} = \frac{e^{h\lambda} - Q(h\lambda)}{h} y_j. \tag{2}$$

The difference between the exponential function and amplification factor can be computed as follows. The Taylor series of $e^{h\lambda}$ with known point 0 is:

$$e^{h\lambda} = 1 + \lambda h + \frac{(\lambda h)^2}{2} + \mathcal{O}(h^3). \tag{3}$$

The Taylor series of $\frac{1}{1-\frac{h}{2}\lambda}$ with known point 0 is:

$$\frac{1}{1-\frac{h}{2}\lambda} = 1 + \frac{1}{2}h\lambda + \frac{1}{4}h^2\lambda^2 + \mathcal{O}(h^3). \tag{4}$$

With (4) it follows that $\frac{1+\frac{h}{2}\lambda}{1-\frac{h}{2}\lambda}$ is equal to

$$\frac{1+\frac{h}{2}\lambda}{1-\frac{h}{2}\lambda} = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \mathcal{O}(h^3). \tag{5}$$

In order to determine $e^{h\lambda} - Q(h\lambda)$, we subtract (5) from (3). Now it follows that

$$e^{h\lambda} - Q(h\lambda) = \mathcal{O}(h^3). \tag{6}$$

The local truncation error can be found by substituting (6) into (2), which leads to

$$\tau_{j+1} = \mathcal{O}(h^2).$$

(c) Application of the trapezoidal rule to

$$y' = -2y + e^t, \text{ with } y(0) = 2,$$

and step size $h = 1$ gives:

$$w_1 = w_0 + \frac{h}{2}[-2w_0 + e^0 - 2w_1 + e].$$

Using the initial value $w_0 = y(0) = 2$ and step size $h = 1$ gives:

$$w_1 = 2 + \frac{1}{2}[-4 - 2w_1 + 1 + e].$$

This leads to

$$2w_1 = 2 + \frac{-3+e}{2} = \frac{1}{2} + \frac{e}{2}, \text{ so } w_1 = \frac{1}{4} + \frac{e}{4}.$$

(d) We use the following definition $x_1 = y$ and $x_2 = y'$. This implies that $x_1' = y' = x_2$ and $x_2' = y'' = -y' - \frac{1}{2}y = -x_2 - \frac{1}{2}x_1$. Writing this in vector notation shows that

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

so $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix}$. To compute the eigenvalues we look for values of λ such that

$$|\mathbf{A} - \lambda\mathbf{I}| = 0.$$

This implies that λ is a solution of

$$\lambda^2 + \lambda + \frac{1}{2} = 0,$$

which leads to the roots:

$$\lambda_1 = -\frac{1}{2} + \frac{1}{2}i \text{ and } \lambda_2 = -\frac{1}{2} - \frac{1}{2}i.$$

(e) To investigate the stability it is sufficient that

$$|Q(h\lambda_1)| \leq 1 \text{ and } |Q(h\lambda_2)| \leq 1.$$

Since λ_1 and λ_2 are complex valued, it is sufficient to check only the first inequality. This leads to

$$\left| \frac{1 + \frac{h(-\frac{1}{2}+\frac{1}{2}i)}{2}}{1 - \frac{h(-\frac{1}{2}+\frac{1}{2}i)}{2}} \right| \leq 1,$$

which is equivalent to

$$\frac{|1 - \frac{h}{4} + \frac{hi}{4}|}{|1 + \frac{h}{4} - \frac{hi}{4}|} \leq 1.$$

Using the definition of the absolute value we arrive at the inequality

$$\frac{\sqrt{(1 - \frac{h}{4})^2 + (\frac{h}{4})^2}}{\sqrt{(1 + \frac{h}{4})^2 + (\frac{h}{4})^2}} \leq 1.$$

This equality is valid for all values of h because

$$\sqrt{(1 - \frac{h}{4})^2 + (\frac{h}{4})^2} \leq \sqrt{(1 + \frac{h}{4})^2 + (\frac{h}{4})^2},$$

for all $h > 0$.

2. (a) The exact answer is 0.25. The composite Trapezoidal rule is given by

$$\frac{1}{2} \cdot \{\frac{1}{2} \cdot 0^3 + (\frac{1}{2})^3 + \frac{1}{2} \cdot 1^3\} = \frac{5}{16} = 0.3125.$$

The difference with the exact answer is $\frac{1}{16} = 0.0625$.

(b) The rounding error is less than

$$h \cdot \{\frac{1}{2}\epsilon + \epsilon \ldots + \epsilon + \frac{1}{2}\epsilon\} \leq n \cdot h \cdot \epsilon = (b - a) \cdot \epsilon.$$

(c) The Taylor polynomial is given by

$$P_1(x) = f(b) + (x - b)f'(b)$$

whereas the truncation error is:

$$f(x) - P_1(x) = \frac{(x-b)^2}{2} f''(\xi), \text{ with } \xi \in [a, b].$$

(d) Integrating this formula gives:

$$\int_a^b P_1(x)dx = \int_a^b f(b) + (x-b)f'(b)dx = (b-a)f(b) - \frac{(a-b)^2}{2} f'(b).$$

Suppose that $M_2 = \max_{\xi \in [a,b]} |f''(\xi)|$. This implies that $|f(x) - P_1(x)| \leq \frac{(x-b)^2}{2} M_2$. Integrating this formula gives:

$$|\int_a^b f(x)dx - \left((b-a)f(b) - \frac{(a-b)^2}{2} f'(b)\right) \leq \int_a^b |f(x) - P_1(x)|dx \leq$$

$$\int_a^b \frac{(x-b)^2}{2} M_2 dx = \frac{(b-a)^3}{6} M_2$$

(e) The composite rule is:

$$h \cdot \{f(a+h) - \frac{h}{2} f'(a+h) + f(a+2h) - \frac{h}{2} f'(a+2h) \ldots + f(b) - \frac{h}{2} f'(b)\}.$$

The result with the composite rule is:

$$\frac{1}{2} \cdot \{(\frac{1}{2})^3 - 3 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot (\frac{1}{2})^2 + (1^3) - 3 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot 1^2\} = \frac{3}{32} = 0.0938.$$

The difference with the exact answer is $\frac{5}{32} = 0.1562$.

(f) For the comparison we note that

- the new method has a worse behavior with respect to rounding errors, because rounding errors of f' also play a role.
- the new method costs n function evaluations (of f') more than the Trapezoidal rule
- The truncation error of the new method is given by

$$\frac{n \cdot h^3}{6} \max_{\xi \in [a,b]} |f''(\xi)| = \frac{(b-a)h^2}{6} \max_{\xi \in [a,b]} |f''(\xi)|$$

  which is 2 times as large as the truncation error of the Trapezoidal rule.

Conclusion: the new method is worse than the Trapezoidal rule.