DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Tuesday 30 January 2007, 9:00-12:00

1. (a) The amplification factor can be derived as follows. Consider the test equation $y' = \lambda y$. Application of the trapezoidal rule to this equation gives:

$$w_{j+1} = w_j + \frac{h}{2}\left(\lambda w_j + \lambda w_{j+1}\right) \tag{1}$$

Rearranging of w_{j+1} and w_j in (1) yields

$$\left(1 - \frac{h}{2}\lambda\right) w_{j+1} = \left(1 + \frac{h}{2}\lambda\right) w_j.$$

It now follows that

$$w_{j+1} = \frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda} w_j,$$

and thus

$$Q(h\lambda) = \frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda}.$$

(b) The definition of the local truncation error is

$$\tau_{j+1} = \frac{y_{j+1} - Q(h\lambda) y_j}{h}.$$

The exact solution of the test equation is given by

$$y_{j+1} = e^{h\lambda} y_j.$$

Combination of these results shows that the local truncation error of the test equation is determined by the difference between the exponential function and the amplification factor $Q(h\lambda)$

$$\tau_{j+1} = \frac{e^{h\lambda} - Q(h\lambda)}{h} y_j. \tag{2}$$

The difference between the exponential function and amplification factor can be computed as follows. The Taylor series of $e^{h\lambda}$ with known point 0 is:

$$e^{h\lambda} = 1 + \lambda h + \frac{(\lambda h)^2}{2} + \mathcal{O}(h^3). \tag{3}$$

The Taylor series of $\frac{1}{1-\frac{h}{2}\lambda}$ with known point 0 is:

$$\frac{1}{1-\frac{h}{2}\lambda} = 1 + \frac{1}{2}h\lambda + \frac{1}{4}h^2\lambda^2 + \mathcal{O}(h^3). \tag{4}$$

With (4) it follows that $\frac{1+\frac{h}{2}\lambda}{1-\frac{h}{2}\lambda}$ is equal to

$$\frac{1+\frac{h}{2}\lambda}{1-\frac{h}{2}\lambda} = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \mathcal{O}(h^3). \tag{5}$$

In order to determine $e^{h\lambda} - Q(h\lambda)$, we subtract (5) from (3). Now it follows that

$$e^{h\lambda} - Q(h\lambda) = \mathcal{O}(h^3). \tag{6}$$

The local truncation error can be found by substituting (6) into (2), which leads to

$$\tau_{j+1} = \mathcal{O}(h^2).$$

(c) Application of the trapezoidal rule to

$$y' = -4y + 2t, \text{ with } y(0) = 2,$$

and step size $h = 1$ gives:

$$w_1 = w_0 + \frac{h}{2}[-4w_0 + 0 - 4w_1 + 2].$$

Using the initial value $w_0 = y(0) = 2$ and step size $h = 1$ gives:

$$w_1 = 2 + \frac{1}{2}[-8 - 4w_1 + 2].$$

This leads to

$$3w_1 = 2 - 3, \text{ so } w_1 = \frac{-1}{3}.$$

(d) We use the following definition $x_1 = y$ and $x_2 = y'$. This implies that $x_1' = y' = x_2$ and $x_2' = y'' = -2y' - 2y = -2x_2 - 2x_1$. Writing this in vector notation shows that

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

so $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$. To compute the eigenvalues we look for values of λ such that

$$|\mathbf{A} - \lambda\mathbf{I}| = 0.$$

This implies that λ is a solution of

$$\lambda^2 + 2\lambda + 2 = 0,$$

which leads to the roots:

$$\lambda_1 = -1 + i \text{ and } \lambda_2 = -1 - i.$$

(e) To investigate the stability it is sufficient that

$$|Q(h\lambda_1| \leq 1 \text{ and } |Q(h\lambda_2| \leq 1.$$

Since λ_1 and λ_2 are complex valued, it is sufficient to check only the first inequality. This leads to

$$\left| \frac{1 + \frac{h(-1+i)}{2}}{1 - \frac{h(-1+i)}{2}} \right| \leq 1,$$

which is equivalent to

$$\frac{|1 - \frac{h}{2} + \frac{hi}{2}|}{|1 + \frac{h}{2} + \frac{hi}{2}|} \leq 1.$$

Using the definition of the absolute value we arrive at the inequality

$$\frac{\sqrt{(1-\frac{h}{2})^2 + (\frac{h}{2})^2}}{\sqrt{(1+\frac{h}{2})^2 + (\frac{h}{2})^2}} \leq 1.$$

This equality is valid for all values of h because

$$\sqrt{(1-\frac{h}{2})^2 + (\frac{h}{2})^2} \leq \sqrt{(1+\frac{h}{2})^2 + (\frac{h}{2})^2},$$

for all $h > 0$.

2.

a) For the Dirichlet problem (1) we use a grid coinciding with the end points $x = 0$ and $x = 1$. For $h = \frac{1}{3}$, we have the two internal points $x_1 = \frac{1}{3}$ and $x_2 = \frac{2}{3}$. At each of these two points the second derivative in the differential equation is replaced by a second order (divided) difference:

$$-\frac{w_{i-1} - 2w_i + w_{i+1}}{h^2} + w_i^2 = 1, \ i = 1, 2.$$

After substitution of the boundary values $w_0 = 0$ and $w_3 = 0$, multiplication by h^2 and substitution of $h = \frac{1}{3}$ we find

$$2w_1 - w_2 + \frac{1}{9}w_1^2 - \frac{1}{9} = 0, \quad (7)$$

$$-w_1 + 2w_2 + \frac{1}{9}w_2^2 - \frac{1}{9} = 0. \quad (8)$$

b) Substitution of $w_1 = w_2 = w$ into (7) yields

$$f(w) = w + \frac{1}{9}w^2 - \frac{1}{9} = 0. \tag{9}$$

Substitution into (8) yields the same equation, showing that the system (7) and (8) admits a symmetric solution. The roots of the quadratic equation (9) are

$$\frac{-9 \pm \sqrt{85}}{2}.$$

The positive root corresponds to the positive solution of the boundary value problem.

c) Using second order interpolation, we have, as an approximation for $y(\frac{1}{2})$:

$$w_{max} = \Sigma_{k=0}^{2} w_k L_k(\frac{1}{2}) = w[L_1(\frac{1}{2}) + L_2(\frac{1}{2})], \tag{10}$$

using $w_0 = 0,\ w_1 = w_2 = w$.
The two Lagrangian polynomials used in this expression are given by:

$$\begin{aligned} L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-\frac{2}{3})}{(\frac{1}{3}-0)(\frac{1}{3}-\frac{2}{3})} = -9x(x-\frac{2}{3}) \\ L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-\frac{1}{3})}{(\frac{2}{3}-0)(\frac{2}{3}-\frac{1}{3})} = \frac{9}{2}x(x-\frac{1}{3}) \end{aligned}$$

Substituting $L_1(\frac{1}{2}) = \frac{3}{4}$ and $L_2(\frac{1}{2}) = \frac{3}{8}$ into (10) gives $\frac{9}{8}w = \frac{9}{16}(-9+\sqrt{85}) \approx 0.123493$.

d) The truncation error $(TE)_i$ for point i is defined by

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} + (TE)_i. \tag{11}$$

The expression for the truncation error $(TE)_i$ follows by expanding the second order difference into Taylor series,

$$(TE)_i = y_i'' - \frac{[y_i - hy_i' + \frac{h^2}{2}y_i'' - \frac{h^3}{6}y_i''' + \frac{h^4}{24}y''''(\xi_{i1})] - 2y_i + [y_i + hy_i' + \frac{h^2}{2}y_i'' + \frac{h^3}{6}y_i''' + \frac{h^4}{24}y''''(\xi_{i2})]}{h^2} \tag{12}$$

$$= -\frac{h^2}{24}[y''''(\xi_{i1}) + y''''(\xi_{i2})]. \tag{13}$$

Because of the intermediate value theorem, $y''''(\xi_{i1}) + y''''(\xi_{i2})$ can be replaced by $2y''''(\eta_i)$, and so

$$(TE)_i = ch^2 y''''(\eta_i), \tag{14}$$

with $c = -\frac{1}{12}$.

e) Replacing y_i'' by (11), two equations for the exact solutions y_1 and y_2 result:

$$-\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - (TE)_i + y_i^2 = 1, \;\; i = 1, 2.$$

Use of the boundary conditions and multiplication by $h^2 = \frac{1}{9}$ yields:

$$2y_1 - y_2 + \frac{1}{9}y_1^2 - \frac{1}{9} = \frac{1}{9}(TE)_1 \tag{15}$$

$$-y_1 + 2y_2 + \frac{1}{9}y_2^2 - \frac{1}{9} = \frac{1}{9}(TE)_2 \tag{16}$$

Using (14), these equations become:

$$2y_1 - y_2 + \frac{1}{9}y_1^2 - \frac{1}{9} = -\frac{1}{12}(\frac{1}{3})^4 y''''(\eta_1), \quad \eta_1 \in (0, \frac{2}{3}) \tag{17}$$

$$-y_1 + 2y_2 + \frac{1}{9}y_2^2 - \frac{1}{9} = -\frac{1}{12}(\frac{1}{3})^4 y''''(\eta_2), \quad \eta_2 \in (\frac{1}{3}, 1) \tag{18}$$

Because the exact solution is symmetric, the solution of this system must satisfy $y_1 = y_2$. The substitution of $y_1 = y_2$ yields identical left hand sides of (17) and (18). For consistency, it is required that $y''''(\eta_1) = y''''(\eta_2)$ which, in view of the shape of the solution curve, demands that η_1 and η_2 are located symmetrically with respect to the point $x = \frac{1}{2}$. The conclusion is that y satisfies the equation

$$f(y) = y + \frac{1}{9}y^2 - \frac{1}{9} = \epsilon = -\frac{1}{12}(\frac{1}{3})^4 y''''(\eta), \tag{19}$$

where η is some point of the interval $(0, 1)$. This equation is the exact counterpart of the equation (9) for the numerical value w.

f) Subtract (9) from (19):

$$f(y) - f(w) = \epsilon,$$

Use the Taylor expansion $f(y) = f(w) + (y - w)\frac{\partial f}{\partial y}(w) + \ldots$, to linearize:

$$f(y) - f(w) = (y - w)\frac{\partial f}{\partial y}(w) = \epsilon.$$

Differentiating the function f,

$$\frac{\partial f}{\partial y}(y) = 1 + \frac{2}{9}y,$$

we can estimate:

$$|y - w| \leq \frac{|\epsilon|}{1 + \frac{2}{9}w} < |\epsilon|,$$

because $w > 0$.
Use of the upper bound $|y''''| < \frac{3}{4}$, as given by property (C), yields

$$|\epsilon| \leq \frac{1}{12}(\frac{1}{3})^4 \max_{0<\xi<1} |y''''(\xi)| < \frac{1}{16 * 81} < 0.0008,$$

from which

$$|y - w| < 0.0008 \tag{20}$$

follows as an upper bound for the errors in the interpolation data.

g) Subtracting (10) from $\overline{w}_{max} = \Sigma_{k=0}^{2} y_k L_k(\frac{1}{2})$,

$$\overline{w}_{max} - w_{max} = \Sigma_{k=0}^{2}(y_k - w_k)L_k(\frac{1}{2}) = (y - w)[L_1(\frac{1}{2}) + L_2(\frac{1}{2})],$$

the estimate

$$|\overline{w}_{max} - w_{max}| \leq |y - w|(\frac{3}{4} + \frac{3}{8}),$$

follows by using the values for the Lagrangian polynomials as computed under c).
Inserting (20), the upper bound

$$|\overline{w}_{max} - w_{max}| < \frac{9}{8} * 0.0008 = 0.0009 \tag{21}$$

for the (absolute value of the) inherent error follows.

h) The interpolation error at $x = \frac{1}{2}$ is given by

$$\frac{(x - x_0)(x - x_1)(x - x_2)}{3!} y'''(\xi) = \frac{(\frac{1}{2} - 0)(\frac{1}{2} - \frac{1}{3})(\frac{1}{2} - \frac{2}{3})}{3!} y'''(\xi) = \frac{1}{432} y'''(\xi).$$

Use of the given upper bound $|y'''| < \frac{1}{8}$ (see (C)) gives

$$|y(\frac{1}{2}) - \overline{w}_{max}| < \frac{1}{432}\frac{1}{8} = 0.00028... < 0.0003 \tag{22}$$

as an upper bound for the interpolation error.

i) The total error $y(\frac{1}{2}) - w_{max}$ is estimated by

$$|y(\frac{1}{2}) - w_{max}| \leq |y(\frac{1}{2}) - \overline{w}_{max}| + |\overline{w}_{max} - w_{max}| < 0.0009 + 0.0003 = 0.0012,$$

using the results (21) and (22). The actual error is

$$y(\frac{1}{2}) - w_{max} = 0.123598625 - 0.12349375 \approx 0.0001,$$

well within the estimated upper bound 0.0012.