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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Wednesday August 29 2007, 14:00-17:00

1. (a) Replace $f(t, y)$ by λy in the RK_4 formulas:

$$\begin{aligned} k_1 &= h\lambda w_n \\ k_2 &= h\lambda(w_n + \tfrac{1}{2}k_1) = h\lambda(1 + \tfrac{1}{2}h\lambda)w_n \\ k_3 &= h\lambda(w_n + \tfrac{1}{2}k_2) = h\lambda(1 + \tfrac{1}{2}h\lambda(1 + \tfrac{1}{2}h\lambda))w_n \\ k_4 &= h\lambda(w_n + k_3) = h\lambda(1 + h\lambda(1 + \tfrac{1}{2}h\lambda(1 + \tfrac{1}{2}h\lambda))w_n \end{aligned}$$

Substitution of these expressions into:

$$w_{n+1} = w_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

and collecting like powers of $h\lambda$ yields:

$$w_{n+1} = [1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4]w_n.$$

The amplification factor is therefore:

$$Q(h\lambda) = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4.$$

(b) The local truncation error is defined as

$$\tau_{n+1} = \frac{y(t_{n+1}) - \overline{w_{n+1}}}{h}, \qquad (1)$$

where $\overline{w_{n+1}}$ is the numerical solution at t_{n+1}, obtained by starting from the exact value $y(t_n)$ in stead of w_n. Repeating the derivation under (a), with w_n replaced by $y(t_n)$, gives:

$$\overline{w_{n+1}} = Q(h\lambda)y(t_n).$$

Using furthermore $y(t_{n+1}) = e^{h\lambda}y(t_n)$ in (1) it follows that

$$\tau_{n+1} = \frac{e^{h\lambda} - Q(h\lambda)}{h}y(t_n).$$

Canceling the first five terms of the expansion of $e^{h\lambda}$ against $Q(h\lambda)$, the required order of magnitude of τ_{n+1} follows.

(c) Use the transformation:

$$\begin{aligned} y_1 &= y\ , \\ y_2 &= y'\ , \end{aligned}$$

This implies that

$$\begin{aligned} y_1' &= y' = y_2\ , \\ y_2' &= y'' = -qy_1 - py_2 + \cos t\ , \end{aligned}$$

So the matrix $\mathbf{A}$ and vector $\mathbf{g}$ are:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix};\ \ \mathbf{g}(t) = \begin{pmatrix} 0 \\ \cos t \end{pmatrix}.$$

Characteristic equation: $\lambda^2 + p\lambda + q = 0$. $\lambda_{1,2} = \frac{-p\pm\sqrt{p^2-4q}}{2}$.

(d) Substitution of the values of p and q into the matrix $\mathbf{A}$ yields the eigenvalues $\lambda_{1,2} = -500 \pm i$. From the given drawing of the stability region the following can be inferred. Because the imaginary part is much smaller than the real part, an approximate stability condition can be obtained by simply neglecting the imaginary part. Then $h \leq 2.8/500$ follows as the stability condition.

(e)

$$y'' + py' + qy = \cos t, \quad y(0) = y_0,\ y'(0) = y_0'. \tag{2}$$

The general solution of (2) is the sum of a homogeneous part, governed by the eigenvalues, and the so called particular solution, which is some linear combination of $\sin t$ and $\cos t$. Both exponentials in the homogeneous part are damped very rapidly because of the (in absolute value) large real part of the eigenvalues and, after a short time of order 10^{-3}, the solution becomes practically equal to the 'smooth' particular solution (time scale of order 1). The smooth solution can be integrated accurately by RK_4 with a 'large' stepsize: a step size of 0.1, let us say, would give an error of order 10^{-4} which is sufficient for most engineering purposes. However stability, governed by the eigenvalues, requires that the stepsize be restricted (see part (d)) to 0.0056. So the stability requirement forces us to choose a stepsize yielding an unnecessarily accurate solution, which is inefficient.

The Trapezoidal rule, on the other hand, is stable for all stepsizes. So the stepsize is restricted by accuracy requirements only. The Trapezoidal rule has a global error of order h^2 such that a good accuracy may be expected for stepsizes of about 0.01, larger than the restriction 0.0056. An efficiency gain may be obtained in spite of the extra work connected with the implicitness of the method.

2. a Let $x_k = kh$ and $y_k := y(x_k)$, then the first order derivative can be approximated by

$$\frac{y_{k+1} - y_{k-1}}{2h} =$$

$$= \frac{y(x_k) + hy'(x_k) + \frac{h^2}{2}y''(x_k) + \frac{h^3}{3!}y'''(\xi_1) - (y(x_k) - hy'(x_k) + \frac{h^2}{2}y''(x_k) - \frac{h^3}{3!}y'''(\xi_2))}{2h}, \tag{3}$$

for a $\xi_1 \in (x_k, x_k + h)$ and $\xi_2 \in (x_k - h, x_k)$. The above expression is rearranged to

$$\frac{y_{k+1} - y_{k-1}}{2h} = y'(x_k) + O(h^2). \tag{4}$$

We use $\frac{y_{k-1} - 2y_k + y_{k+1}}{h^2}$ to approximate the second derivative. Using Taylor expansions, one obtains with $\xi_1 \in (x_{k-1}, x_k)$ and $\xi_2 \in (x_k, x_{k+1})$

$$\begin{aligned} y_{k-1} &= y(x_k - h) = y(x_k) - hy'(x_k) + \frac{h^2}{2!}y''(x_k) - \frac{h^3}{3!}y'''(x_k) + \frac{h^4}{4!}y''''(\xi_1), \\ y_{k+1} &= y(x_k + h) = y(x_k) + hy'(x_k) + \frac{h^2}{2!}y''(x_k) + \frac{h^3}{3!}y'''(x_k) + \frac{h^4}{4!}y''''(\xi_2). \end{aligned} \tag{5}$$

Substitution of these expressions into the approximation for the second derivative, gives

$$\frac{-y_{k-1} + 2y_k - y_{k+1}}{h^2} = y''(x_k) + O(h^2). \tag{6}$$

Further, we have Dirichlet conditions, hence the above equation holds for all $i \in \{1, \ldots, n\}$. Using the approximations for the first- and second order derivative, gives the following discretization of the given boundary value problem

$$-x_k^2 \frac{w_{k+1} - 2w_k + w_{k-1}}{h^2} + x_k \frac{w_{k+1} - w_{k-1}}{2h} + w_k = x_k^2, \text{ for } i \in \{1, \ldots, n\}. \tag{7}$$

The local truncation error is given by

$$\begin{aligned} &(\underline{\varepsilon})_k = (A\underline{y} - \underline{f})_k = -x_k^2 \frac{y_{k-1} - 2y_k + y_{k+1}}{h^2} + x_k \frac{y_{k+1} - y_{k-1}}{2h} + y_k - x_k^2 = \\ &= -x_k^2 \frac{y_{k-1} - 2y_k + y_{k+1}}{h^2} + x_k \frac{y_{k+1} - y_{k-1}}{2h} + y_k - \left[-x_k^2 y''(x_k) + x_k y'(x_k) + y(x_k)\right] = \\ &= O(h^2). \end{aligned} \tag{8}$$

In which the differential equation has been substituted to arrive at the second equality of the above equation.

b For $n = 2$, we have $h = 1/3$. Elaboration of equation (5) for the two internal grid nodes, gives

$$-x_1^2 \frac{w_2 - 2w_1 + w_0}{h^2} + x_1 \frac{w_2 - w_0}{2h} + w_1 = x_1^2,$$
$$-x_2^2 \frac{w_3 - 2w_2 + w_1}{h^2} + x_2 \frac{w_3 - w_1}{2h} + w_2 = x_2^2. \tag{9}$$

Substitution of $h = 1/3$, $x_1 = 1/3$, $x_2 = 2/3$, $y_0 = 0$ and $y_3 = 1$, gives

$$-(w_2 - 2w_1) + 1/2w_2 + w_1 = 1/9,$$
$$-4(1 - 2w_2 + w_1) + 1 - w_1 + w_2 = 4/9. \tag{10}$$

Rearrangement of the above equations, gives

$$3w_1 - 1/2w_2 = 1/9,$$
$$-5w_1 + 9w_2 = 4/9 + 4 - 1 = 31/9. \tag{11}$$

c i The Trapezoidal Rule for integration is given by

$$\int_a^b f(x)dx \approx \frac{b-a}{2}(f(a) + f(b)). \tag{12}$$

Let $x_k = a + kh$ and $x_n = b$, then the composite Trapezoidal Rule is given by

$$\int_a^b f(x)dx \approx h\left(\frac{f(a)}{2} + f(x_1) + \ldots + f(x_{n-1}) + \frac{f(b)}{2}\right). \tag{13}$$

ii The approximation for the integral using the composite Trapezoidal Rule with $h = 1/3$ is

$$\int_0^1 y(x)dx \approx \frac{1}{3}(1/9 + 4/9 + 1/2) = 19/54. \tag{14}$$

d i Given two points $(x_0, y(x_0))$ and $(x_1, y(x_1))$, then the linear interpolation polynomial, $P(x)$, is given by

$$P(x) = y(x_0)\frac{x - x_1}{x_0 - x_1} + y(x_1)\frac{x - x_0}{x_1 - x_0}. \tag{15}$$

Substitution of the points $x_0 = 2/3$, $x_1 = 1$, $y(x_0) = 4/9$ and $y(x_0) = 4/9$ and $y(1) = 1$, yields

$$P(x) = 4/9 \cdot \frac{x - 1}{2/3 - 1} + 1 \cdot \frac{x - 2/3}{1 - 2/3} = 5/3x - 2/3. \tag{16}$$

ii Next, we find $\tilde{x}$ such that $P(\tilde{x}) = 1/2$, hence, we solve

$$5/3\tilde{x} - 2/3 = 1/2 \Longleftrightarrow \tilde{x} = 7/10. \tag{17}$$