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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Tuesday April 3 2007, 14:00-17:00

1. (a) The local truncation error is given by

$$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h}. \tag{1}$$

Here we obtain y_{n+1} by a Taylor expansion around t_n:

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2} y''(t_n) + O(h^3). \tag{2}$$

From the Chain Rule of Differentiation, we know

$$\begin{aligned} y'(t_n) &= f(t_n, y_n) \\ y''(t_n) &= \frac{df(t_n, y_n)}{dt} = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} y'(t_n) = \\ &= \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} f(t_n, y_n). \end{aligned} \tag{3}$$

Hence, one obtains

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2} \left(\frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} f(t_n, y_n) \right) + O(h^3). \tag{4}$$

For z_{n+1}, we obtain, after substitution of the predictor step for z^*_{n+1} into the corrector step and after a Taylor expansion around (t_n, y_n)

$$z_{n+1} = y_n + h\left((1-\theta) f(t_n, y_n) + \theta f(t_n + h, y_n + h f(t_n, y_n))\right) =$$

$$y_n + h \left((1-\theta) f(t_n, y_n) + \theta (f(t_n, y_n) + h(\frac{\partial f(t_n, y_n)}{\partial t} + f(t_n, y_n) \frac{\partial f(t_n, y_n)}{\partial y})) + O(h^2) \right). \tag{5}$$

Subsequently, it follows that

$$y_{n+1} - z_{n+1} = O(h^2), \text{ and, hence } \tau_{n+1}(h) = \frac{O(h^2)}{h} = O(h) \text{ for } 0 \le \theta \le 1, \tag{6}$$

$$y_{n+1} - z_{n+1} = O(h^3), \text{ and, hence } \tau_{n+1}(h) = \frac{O(h^3)}{h} = O(h^2) \text{ for } \theta = \frac{1}{2}. \tag{7}$$

(b) Consider the test equation $y' = \lambda y$, then, herewith, one obtains

$$\begin{aligned} &w_{n+1} = w_n + h\lambda w_n = (1+h\lambda)w_n, \\ &w_{n+1} = w_n + h((1-\theta)\lambda w_n + \theta\lambda w^*_{n+1}) = \\ &= w_n + h((1-\theta)\lambda w_n + \theta\lambda(w_n + h\lambda w_n)) = (1 + h\lambda + \theta(h\lambda)^2)w_n. \end{aligned} \tag{8}$$

Hence the amplification factor is given by

$$Q(h\lambda) = 1 + h\lambda + \theta(h\lambda)^2. \tag{9}$$

(c) Let $\underline{w}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Application of the numerical method, yields for the predictor

$$\underline{w}^*_1 = \underline{w}_0 + h \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \underline{w}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}. \tag{10}$$

For the corrector step, we get with $\theta = \frac{1}{2}$

$$\underline{w}_1 = \underline{w}_0 + \frac{h}{2} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} (\underline{w}_0 + \underline{w}^*_1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{11}$$

(d) To analyse numerical stability, the eigenvalues of the matrix A are needed. These eigenvalues are given by $\lambda_{1,2} = -1 \pm i$. Subsequently, the eigenvalues are substituted into the amplification factor from assignment b. Then, one obtains with $h = 1$

$$Q(h\lambda) = 1 + (-1+i) + \theta(-1+i)^2 = i(1-2\theta). \tag{12}$$

Hence, for the modulus of the amplification factor, we obtain

$$|Q(h\lambda)| = |1 - 2\theta|, \tag{13}$$

This implies that the method is stable for $h = 1$ for all $0 \le \theta \le 1$. This holds for both eigenvalues.

(e) The amplification factor for the backward method due to Euler follows from the test equation:

$$w_{n+1} = w_n + h\lambda w_{n+1} \Leftrightarrow w_{n+1} = \frac{1}{1-h\lambda}.. \tag{14}$$

With $\lambda = -1 \pm i$, we obtain

$$Q(h\lambda) = \frac{1}{1 - h(-1 \pm i)} = \frac{1}{1 + h \mp ih}. \tag{15}$$

This implies

$$|Q(h\lambda)|^2 = \frac{1}{(1+h)^2 + h^2} < 1 \text{ for all } h > 0. \tag{16}$$

This implies that the backward Euler method is stable for each $h > 0$.

2. (a) Taylor polynomials are:

$$\begin{aligned} f(0) &= f(0)\,, \\ f(h) &= f(0) + hf^{'}(0) + \frac{h^2}{2}f^{''}(0) + \frac{h^3}{6}f^{'''}(\xi_1)\,, \\ f(2h) &= f(0) + 2hf^{'}(0) + 2h^2 f^{''}(0) + \frac{(2h)^3}{6}f^{'''}(\xi_2)\,. \end{aligned}$$

We know that $Q(h) = \alpha_0 f(0) + \alpha_1 f(h) + \alpha_2 f(2h)$, which should be equal to $f''(0) + O(h)$. This leads to the following conditions:

$$\begin{array}{lcccccccc} f(0): & \alpha_0 & + & \alpha_1 & + & \alpha_2 & = & 0\,, \\ f^{'}(0): & & & h\alpha_1 & + & 2h\alpha_2 & = & 0\,, \\ f^{''}(0): & & & \frac{h^2}{2}\alpha_1 & + & 2h^2\alpha_2 & = & 1\,. \end{array}$$

(b) The truncation error follows from the Taylor polynomials:

$$f''(0) - Q(h) = f''(0) - \frac{f(0) - 2f(h) + f(2h)}{h^2} = \frac{\frac{-2h^3}{6}f'''(\xi_1) + \frac{8h^3}{6}f'''(\xi_2)}{h^2},$$

$$= hf'''(\xi).$$

(c) Note that

$$f''(0) - Q(h) = Kh \tag{17}$$

$$f''(0) - Q(\frac{h}{2}) = K(\frac{h}{2}) \tag{18}$$

Subtraction gives:

$$Q(\frac{h}{2}) - Q(h) = Kh - K(\frac{h}{2}) = K(\frac{h}{2}) \tag{19}$$

We choose $h = \frac{1}{2}$. Then $Q(h) = Q(\frac{1}{2}) = \frac{0 - 2\times 0.4794 + 0.8415}{(\frac{1}{2})^2} = -0.4692$ and $Q(\frac{h}{2}) = Q(\frac{1}{4}) = \frac{0 - 2\times 0.2474 + 0.4794}{(\frac{1}{4})^2} = -0.2464$. Combining (18) and (19) shows that

$$f''(0) - Q(\frac{1}{4}) = Q(\frac{1}{4}) - Q(\frac{1}{2}) = 0.2228.$$

(d) Since only 4 digits are given the rounding error is: $\epsilon = 0.00005$.

To estimate the rounding error we note that

$$|Q(h) - \hat{Q}(h)| = |\frac{(f(0) - \hat{f}(0)) - 2(f(h) - \hat{f}(h)) + (f(2h) - \hat{f}(2h))}{h^2}|$$

$$\leq \frac{|f(0) - \hat{f}(0)| + 2|f(h) - \hat{f}(h)| + |f(2h) - \hat{f}(2h)|}{h^2} = \frac{4\epsilon}{h^2},$$

so $C_1 = 4$.

(e) The total error is bounded by

$$\begin{aligned}|f''(0) - \hat{Q}(h)| &= |f''(0) - Q(h) + Q(h) - \hat{Q}(h)| \\ &\leq |f''(0) - Q(h)| + |Q(h) - \hat{Q}(h)| \\ &\leq \frac{1}{3}h + \frac{4\epsilon}{h^2} = g(h)\end{aligned}$$

This is minimal if $g'(h) = 0$. Note that $g'(h) = \frac{1}{3} - \frac{8\epsilon}{h^3}$. This implies that $h_{opt}^3 = 24 \cdot 0.00005$, so $h_{opt} = 0.0012^{\frac{1}{3}} = 0.1063$.