DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Wednesday 30 August 2006, 9:00-12:00

1.

a The local truncation error is defined as

$$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h}, \tag{1}$$

where z_{n+1} is given by

$$z_{n+1} = y_n + h\left(a_1 f(t_n, y_n) + a_2 f(t_n + h, y_n + hf(t_n, y_n))\right). \tag{2}$$

A Taylor expansion of f around (t_n, y_n) yields

$$f(t_n+h, y_n+hf(t_n, y_n)) = f(t_n, y_n)+h\frac{\partial f}{\partial t}(t_n, y_n)+hf(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n)+O(h^2). \tag{3}$$

This is substituted into equation (2) to obtain

$$z_{n+1} = y_n+h\left(a_1 f(t_n, y_n) + a_2\left[f(t_n, y_n) + h\frac{\partial f}{\partial t}(t_n, y_n) + hf(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n)\right]\right)+O(h^3). \tag{4}$$

A Taylor series for $y(x)$ around t_n gives for y_{n+1}

$$y_{n+1} = y(t_n + h) = y_n + hy'(t_n) + \frac{h^2}{2}y''(t_n) + O(h^3). \tag{5}$$

Equations (5) and (4) are substituted into relation (1) to obtain

$$\tau_{n+1}(h) = f(t_n, y_n)(1 - (a_1 + a_2)) + h\left(\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right)\left(\frac{1}{2} - a_2\right) + O(h^2) \tag{6}$$

Hence

(a) $a_1 + a_2 = 1$ implies $\tau_{n+1}(h) = O(h)$;

(b) $a_1 + a_2 = 1$ and $a_2 = 1/2$, that is, $a_1 = a_2 = 1/2$, gives $\tau_{n+1}(h) = O(h^2)$.

b The auxiliary equation is given by

$$y' = \lambda y. \tag{7}$$

Application of the predictor step to the auxiliary equation gives

$$w^*_{n+1} = w_n + h\lambda w_n = (1 + h\lambda)w_n. \tag{8}$$

The corrector step yields

$$w_{n+1} = w_n + h\left(a_1\lambda w_n + a_2\lambda(1 + h\lambda)w_n\right) = (1 + (a_1 + a_2)h\lambda + a_2h^2\lambda^2)w_n. \tag{9}$$

Hence the amplification factor is given by

$$Q(h\lambda) = 1 + (a_1 + a_2)h\lambda + a_2h^2\lambda^2. \tag{10}$$

c Let $\lambda < 0$ (so λ is real), then, for stability, the amplification factor must satisfy

$$-1 \leq Q(h\lambda) \leq 1, \tag{11}$$

from the previous assignment, we have

$$-1 \leq 1 + (a_1 + a_2)h\lambda + a_2(h\lambda)^2 \leq 1 \Leftrightarrow -2 \leq (a_1 + a_2)h\lambda + a_2(h\lambda)^2 \leq 0. \tag{12}$$

First, we consider the left inequality:

$$a_2(h\lambda)^2 + (a_1 + a_2)h\lambda + 2 \geq 0 \tag{13}$$

For $h\lambda = 0$, the above inequality is satisfied, further the discriminant is given by $(a_1 + a_2)^2 - 8a_2 < 0$. Here the last inequality follows from the given hypothesis. Hence the left inequality in relation (12) is always satisfied. Next we consider the right hand inequality of relation (12)

$$a_2(h\lambda)^2 + (a_1 + a_2)h\lambda \leq 0. \tag{14}$$

This relation is rearranged into

$$a_2(h\lambda)^2 \leq -(a_1 + a_2)h\lambda, \tag{15}$$

hence

$$a_2|h\lambda|^2 \leq (a_1 + a_2)|h\lambda| \Leftrightarrow |h\lambda| \leq \frac{a_1 + a_2}{a_2}, \qquad a_2 \neq 0. \tag{16}$$

This results into the following condition for stability

$$h \leq \frac{a_1 + a_2}{a_2|\lambda|}, \qquad a_2 \neq 0. \tag{17}$$

d The Jacobian, J, is given by

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} \\ \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} \end{pmatrix}. \tag{18}$$

Since $f_1(y_1, y_2) = -y_1 y_2$ and $f_2(y_1, y_2) = y_1 y_2 - y_2$, we obtain

$$J = \begin{pmatrix} -y_2 & -y_1 \\ y_2 & y_1 - 1 \end{pmatrix}. \tag{19}$$

Substitution of the initial values $y_1(0) = y_2(0) = 1$, gives

$$J = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{20}$$

e The eigenvalues of the Jacobian at $y_1(0) = y_2(0) = 1$ are given by $\lambda_{1,2} = -1/2(1 \pm i\sqrt{3})$. For our case, we have

$$Q(h\lambda) = 1 + h\lambda + 1/2(h\lambda)^2. \tag{21}$$

Since our eigenvalues are not real valued, it is required for stability that

$$|Q(h\lambda)| \leq 1. \tag{22}$$

Since the eigenvalues are complex conjugates, we can proceed with one of the eigenvalues, say $\lambda = -1/2(1 + i\sqrt{3})$ with $\lambda^2 = 1/2(-1 + i\sqrt{3})$, to obtain

$$\begin{aligned} Q(h\lambda) &= 1 + h(-1/2 - 1/2i\sqrt{3}) + 1/2h^2(-1/2 + 1/2i\sqrt{3}) = \\ &= 1 - 1/2h - 1/4h^2 + \tfrac{i\sqrt{3}}{2}(1/2h^2 - h). \end{aligned} \tag{23}$$

Next, we compute the square of the modulus of the above expression, to obtain

$$(1 - 1/2h - 1/4h^2)^2 + \frac{3}{4}(1/2h^2 - h)^2 \leq 1. \tag{24}$$

After some elementary algebra, and division by h, we obtain

$$-1 + \frac{h}{2} - \frac{h^2}{2} + \frac{h^3}{4} \leq 0 \Leftrightarrow h^3 - 2h^2 + 2h \leq 4. \tag{25}$$

It can be seen that the derivative of $h^3 - 2h^2 + 2h$ is positive and never zero, hence $h^3 - 2h^2 + 2h$ is monotonically increasing. Further, by direct substitution it can be seen that $h = 2$ gives equality in relation (25). Hence, the criterion for stability becomes $h \leq 2$.

Remark: The candidate should be rewarded with all the credits minus 0.5 if he or she reaches relation (25).

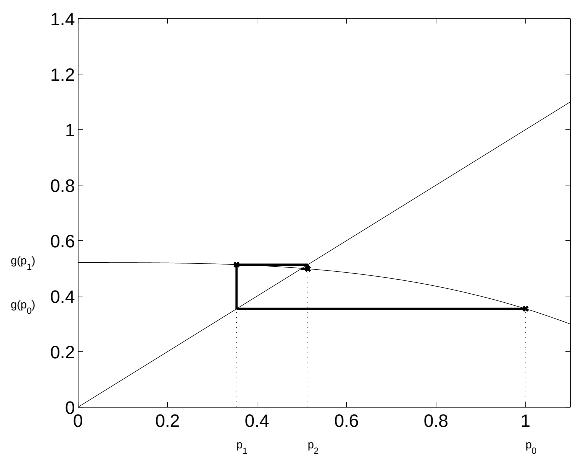
2. (a) A fixed point p satisfies the equation $p = g(p)$. Substitution gives: $p = -\frac{p^3}{6} + \frac{25}{48}$. Rewriting this expression gives:

$$\begin{aligned} \frac{p^3}{6} + p - \frac{25}{48} &= 0 \\ p^2 + 6 - \frac{25}{8p} &= 0 \\ f(p) &= 0 \end{aligned}$$

The fixed point iteration is defined by: $p_{i+1} = g(p_i)$. Starting with $p_0 = 1$ one obtains:

$$\begin{aligned} p_1 &= 0.3542 \\ p_2 &= 0.5134 \\ p_3 &= 0.4983 \end{aligned}$$

(b) The fixed point iteration is illustrated in the next figure.



(c) For the convergence two conditions should be satisfied:

- $g(p) \in [0, 1]$ for all $p \in [0, 1]$.
- $|g'(p)| \leq k < 1$ for all $p \in [0, 1]$.

Since $g(p) = -\frac{p^3}{6} + \frac{25}{48}$ it follows that $g'(p) = -\frac{p^2}{2}$. Note that $g'(p) \leq 0$ for all $p \in [0, 1]$. This implies that $\frac{25}{48} = g(0) \geq g(p) \geq g(1) = \frac{17}{48}$ for all $p \in [0, 1]$, so the first condition holds. For the second condition we note that $|g'(p)| = \frac{p^2}{2} \leq \frac{1}{2} = k < 1$ for all $p \in [0, 1]$, so the second conditions is also satisfied, which implies that the fixed point iteration is convergent for all $p_0 \in [0, 1]$.

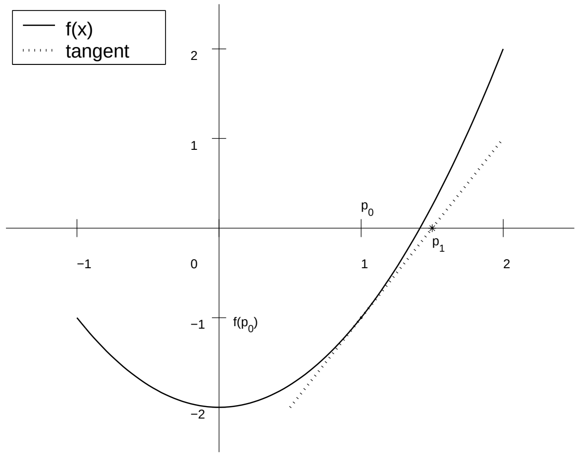


Figure 1: The Newton-Raphson method

(d) Graphically the Newton-Raphson method is given in Figure 1. The tangent in $(p_0, f(p_0))$ is given by:

$$l(x) = f(p_0) + (x - p_0)f'(p_0)$$

Taking $l(p_1) = 0$ leads to

$$f(p_0) + (p_1 - p_0)f'(p_0) = 0$$

Rewriting gives $p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}$.

(e) Starting with $p_0 = 1$ we note that

$$\begin{aligned} f(p) &= p^2 + 6 - \frac{25}{8p}, \quad f(1) = 1 + 6 - \frac{25}{8} = 3\frac{7}{8} \\ f'(p) &= 2p + \frac{25}{8p^2}, \quad f'(1) = 5\frac{1}{8} \end{aligned}$$

Substituting this into the formula gives $p_1 = 1 - \frac{3\frac{7}{8}}{5\frac{1}{8}} = 1 - \frac{31}{41} = \frac{10}{41} = 0.2439$.

(f) One can prove that the convergence of the Newton-Raphson method is quadratically as follows:

$$0 = f(p) = f(p_n) + (p - p_n)f^{'}(p_n) + \frac{(p - p_n)^2}{2}f^{''}(\xi_n)\ , \xi_n \in (p_n, p)\ .$$

Due to the definition one has

$$0 = f(p_n) + (p_{n+1} - p_n)f^{'}(p_n)\ .$$

Subtraction yields

$$p_{n+1} - p = (p_n - p)^2 \frac{f^{''}(\xi)}{2f^{'}(p_n)}\ .$$

which implies quadratic convergence.