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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Tuesday 4 April 2006, 9:00-12:00

1. (a) The local truncation error is given by

$$\tau = \frac{y_{n+1} - z_{n+1}}{h}$$

where z_{n+1} is given by

$$z_{n+1} = y_n + h[\frac{3}{2}f(t_n, y_n) - \frac{1}{2}f(t_{n-1}, y_{n-1})]$$

Due to the differential equation, this can also be written as:

$$z_{n+1} = y_n + h(\frac{3}{2}y'_n - \frac{1}{2}y'_{n-1}). \quad (1)$$

The last term between brackets can also be developed in a Taylor polynomial:

$$y'_{n-1} = y'_n - hy''_n + O(h^2) \quad (2)$$

Substitution of this into (1) yields

$$z_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + O(h^3) \quad (3)$$

Consider the exact solution at t_{n+1} and make a Taylor polynomial:

$$y_{n+1} = y_n + hy'_n + \frac{1}{2}h^2y''_n + O(h^3) \quad (4)$$

Putting this into the definition of the local truncation error shows that the local truncation error is $O(h^2)$.

(b) Application of the method to the test equation shows that:

$$w_{n+1} = w_n + h\{\tfrac{3}{2}\lambda w_n - \tfrac{1}{2}\lambda w_{n-1}\} \quad (5)$$

Since $w_n = Q(h\lambda)w_{n-1}$ and $w_{n+1} = Q(h\lambda)w_n = Q(h\lambda)^2w_{n-1}$ we obtain

$$\{Q(h\lambda)\}^2 - (1 + \tfrac{3}{2}h\lambda)Q(h\lambda) + \tfrac{1}{2}h\lambda\}w_{n-1} = 0 \quad (6)$$

for all values of n. From this the equation follows.

(c) This equation has two roots:

$$Q_1(h\lambda) = \frac{1+\frac{3}{2}h\lambda+\sqrt{1+h\lambda+(\frac{3}{2}h\lambda)^2}}{2} \tag{7}$$
$$Q_2(h\lambda) = \frac{1+\frac{3}{2}h\lambda-\sqrt{1+h\lambda+(\frac{3}{2}h\lambda)^2}}{2}$$

The discriminant is always positive, so both roots are real. It easily follows that $Q_2(h\lambda) < Q_1(h\lambda)$ and both should satisfy $-1 \leq Q(h\lambda) \leq 1$. This implies that: $-1 \leq Q_2(h\lambda)$ and $Q_1(h\lambda) \leq 1$. The second inequality yields:

$$\sqrt{1+h\lambda+(\tfrac{3}{2}h\lambda)^2} \leq 1-\tfrac{3}{2}h\lambda \tag{8}$$
$$1+h\lambda+(\tfrac{3}{2}h\lambda)^2 \leq 1-3h\lambda+(\tfrac{3}{2}h\lambda)^2 \tag{9}$$
$$0 \leq -4h\lambda \tag{10}$$

Since $\lambda < 0$ this inequality always holds for all values of $h > 0$. The first inequality gives:

$$-3-\tfrac{3}{2}h\lambda \leq -\sqrt{1+h\lambda+(\tfrac{3}{2}h\lambda)^2} \tag{11}$$
$$3+\tfrac{3}{2}h\lambda \geq \sqrt{1+h\lambda+(\tfrac{3}{2}h\lambda)^2} \tag{12}$$
$$9+9h\lambda+(\tfrac{3}{2}h\lambda)^2 \geq 1+h\lambda+(\tfrac{3}{2}h\lambda)^2 \tag{13}$$
$$8+8h\lambda \geq 0 \tag{14}$$
$$h \leq \frac{1}{-\lambda} \tag{15}$$

(d) The eigenvalues λ of the matrix

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

are computed by solving the equation:

$$det\begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix} = \lambda^2+3\lambda+2 = 0.$$

Solutions are: $\lambda_1 = -1$ and $\lambda_2 = -2$. So the system of differential equations is stable and the method can be used in a stable way if $h \leq \frac{1}{2}$.

(e) The numerical solution on time step t is given by $\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$ From the initial conditions we obtain:

$$\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Doing one step Euler forward we get:

$$\begin{pmatrix} w_1(h) \\ w_2(h) \end{pmatrix} = \begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} + h[\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}]$$

Putting the values into the equations one obtains:

$$\begin{pmatrix} w_1(h) \\ w_2(h) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2}[\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}] = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

The second step with the new method gives:

$$\begin{pmatrix} w_1(2h) \\ w_2(2h) \end{pmatrix} = \begin{pmatrix} w_1(h) \\ w_2(h) \end{pmatrix} +$$

$$h\left(\frac{3}{2}[\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} w_1(h) \\ w_2(h) \end{pmatrix} + \begin{pmatrix} 0 \\ h \end{pmatrix}] - \frac{1}{2}[\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} w_1(0) \\ w_2(0) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}]\right)$$

Substitution of the numbers gives:

$$\begin{pmatrix} w_1(2h) \\ w_2(2h) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} + \frac{1}{2}\left(\begin{pmatrix} -\frac{3}{4} \\ \frac{3}{2} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ 1\frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} -\frac{1}{8} \\ 1 \end{pmatrix}$$

2. a We use $\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$ to approximate the second derivative and hence the local truncation error is given by

$$\varepsilon(h) := y''(x_i) - \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}. \tag{16}$$

Using Taylor expansions, one obtains with $\xi_1 \in (x_{i-1}, x_i)$ and $\xi_2 \in (x_i, x_{i+1})$

$$\begin{aligned} y_{i-1} &= y(x_i - h) = y(x_i) - hy'(x_i) + \frac{h^2}{2!}y''(x_i) - \frac{h^3}{3!}y'''(x_i) + \frac{h^4}{4!}y''''(\xi_1), \\ y_{i+1} &= y(x_i + h) = y(x_i) + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y'''(x_i) + \frac{h^4}{4!}y''''(\xi_2). \end{aligned} \tag{17}$$

Substitution of these expressions into (16) yields

$$\varepsilon(h) = \frac{h^2}{4!}\{y''''(\xi_1) + y''''(\xi_2)\} = O(h^2). \tag{18}$$

Further, we have Dirichlet conditions, hence the above equation holds for all $i \in \{1, \ldots, n\}$. Using this approximation for the second order derivative, gives the following discretization of the given boundary value problem

$$\frac{-w_{i-1} + 2w_i - w_{i+1}}{h^2} + w_i^2 = \frac{4}{x_i} \left(= \frac{4}{ih} = \frac{4(n+1)}{i}\right), \text{ for } i \in \{1, \ldots, n\}. \tag{19}$$

b For $n = 3$, we have $h = 1/4$, herewith one obtains

$$-16w_{i-1} + 32w_i - 16w_{i+1} + w_i^2 = \frac{16}{i}, \text{ for } i \in \{1, 2, 3\}. \tag{20}$$

Implementation of the boundary conditions $w_0 = 0$ and $w_4 = 1$, gives

$$\begin{cases} 32w_1 - 16w_2 + w_1^2 = 16, \\ -16w_1 + 32w_2 - 16w_3 + w_2^2 = 8, \\ -16w_2 + 32w_3 + w_3^2 = 16/3 + 1/h^2 = 16/3 + 16 = 64/3. \end{cases} \tag{21}$$

c So we have

$$\begin{aligned} f_1(w_1, w_2) &= 18w_1 - 9w_2 + w_1^2 - 9, \\ f_2(w_1, w_2) &= -9w_1 + 18w_2 + w_2^2 - 9/2. \end{aligned} \tag{22}$$

The Jacobian of these functions is given by

$$J(w_1, w_2) = \begin{pmatrix} 18 + 2w_1 & -9 \\ -9 & 18 + 2w_2 \end{pmatrix} \Rightarrow J(3, 0) = \begin{pmatrix} 24 & -9 \\ -9 & 18 \end{pmatrix}. \tag{23}$$

Using Newton Raphson iterations, we first determine $\underline{s}_1$ from

$$J(3, 0)\underline{s}_1 = -\underline{f}(3, 0) \Rightarrow \begin{pmatrix} 24 & -9 \\ -9 & 18 \end{pmatrix} \underline{s}_1 = \begin{pmatrix} -54 \\ 63/2 \end{pmatrix}. \tag{24}$$

From this, one gets $\underline{s}_1 = \begin{pmatrix} -1.9615 \\ 0.7692 \end{pmatrix}$. Then the updated estimate for the solution $\underline{p}_1$ is given by $\underline{p}_1 = \underline{p}_0 + \underline{s}_1 = \begin{pmatrix} 1.0385 \\ 0.7692 \end{pmatrix}$.

d i The linear Lagrangian polynomial is given by

$$p(x) = f(x_0)\frac{x - x_1}{x_0 - x_1} + f(x_1)\frac{x - x_0}{x_1 - x_0}. \tag{25}$$

With $x = 1/4$, $x_0 = 0$, $x_1 = 1/3$, $f(x_0) = 0$ and $f(x_1) = 5$, this gives

$$p(1/4) = 15/4. \tag{26}$$

This is the approximation of $f(1/4)$ using linear interpolation.

ii The linear Lagrangian polynomial with the exact values of the function is given by

$$p(x) = f(x_0)\frac{x - x_1}{x_0 - x_1} + f(x_1)\frac{x - x_0}{x_1 - x_0}. \tag{27}$$

For the measured values of the function we have analogously

$$\tilde{p}(x) = \tilde{f}(x_0)\frac{x - x_1}{x_0 - x_1} + \tilde{f}(x_1)\frac{x - x_0}{x_1 - x_0}. \tag{28}$$

Subtraction of equation (28) from (27) gives

$$|p(x) - \tilde{p}(x)| \leq |f(x_0) - \tilde{f}(x_0)||\frac{x - x_1}{x_0 - x_1}| + |f(x_1) - \tilde{f}(x_1)||\frac{x - x_0}{x_1 - x_0}| =$$

$$= \varepsilon \left\{ |\frac{x - x_1}{x_0 - x_1}| + |\frac{x - x_0}{x_1 - x_0}| \right\}. \tag{29}$$

With the values of x, x_0 and x_1 as defined above, we have

$$|p(x) - \tilde{p}(x)| \leq \varepsilon. \tag{30}$$

Remark: This could also be derived using a graph.