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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Wednesday 31 August 2005, 9:00-12:00

1. (a) Local truncation error $\tau_{j+1}(h)$:

$$\begin{aligned}\tau_{j+1}(h) &:= \frac{y_{j+1}-\overline{w}_{j+1}}{h} = \frac{y_{j+1}-y_j}{h} - (1-\beta)f(t_j,y_j) - \beta f(t_{j+1},\overline{w}^*_{j+1}) = \\ &= \frac{y_{j+1}-y_j}{h} - (1-\beta)f(t_j,y_j) - \beta f(t_{j+1}, y_j + hf(t_j,y_j))\end{aligned} \tag{1}$$

Taylor expansion of y_{j+1} yields:

$$y_{j+1} = y_j + hy'_j + \frac{h^2}{2}y''_j + O(h^3).$$

Furthermore, we can give a Taylor expansion of $f(t_{j+1}, y_j+hf(t_j,y_j))$ as follows:

$$f(t_j+h, y_j+hf(t_j,y_j)) = f(t_j,y_j) + h(f_t)_j + hf(t_j,y_j)(f_y)_j + O(h^2).$$

Substituting these expansions in (1) and (2) shows that:

$$\begin{aligned}\tau_{j+1}(h) &= y'(t_j) + \frac{h}{2}y''(t_j) + O(h^2) - (1-\beta)f(t_j,y_j) + \\ &-\beta\left\{f(t_j,y_j) + hf_t(t_j,y_j) + hf(t_j,y_j)f_y(t_j,y_j) + O(h^2)\right\}.\end{aligned} \tag{2}$$

Since $y' = f(t,y)$, use of the Chain Rule for differentiation gives

$$\begin{aligned}y'(t_j) &= f(t_j,y_j),\\ y''(t_j) &= f_t(t_j,y_j) + f_y(t_j,y_j)f(t_j,y_j),\end{aligned} \tag{3}$$

Hence after substitution into equation (2) we obtain:

$$\tau_{j+1}(h) = \frac{h}{2}\left\{f_t(t_j,y_j) + f_y(t_j,y_j)f(t_j,y_j)\right\} - \beta h\left\{f_t(t_j,y_j) + f_y(t_j,y_j)f(t_j,y_j)\right\} + O(h^2). \tag{4}$$

This implies that $\tau_{j+1}(h) = O(h^2)$ if $\beta = \frac{1}{2}$ and $\tau_{j+1}(h) = O(h)$ if $\beta \neq \frac{1}{2}$.

(b) We consider the amplification factor for the test-equation $y' = \lambda y$, then

$$w_{j+1}^{*} = w_j + h\lambda w_j \tag{5}$$

Hence we have

$$w_{j+1} = w_j + (1-\beta)h\lambda w_j + \beta(h\lambda + h^2\lambda^2)w_j = \tag{6}$$

$$\tag{7}$$

$$= w_j\left\{1 + h\lambda + \beta h^2\lambda^2\right\} \Rightarrow Q(h\lambda) = 1 + h\lambda + \beta h^2\lambda^2. \tag{8}$$

This $Q(h\lambda)$ is the amplification factor we need.

(c) Eigenvalues $\lambda_{1,2} = \pm 2i$. Hence the amplification factor is

$$Q(h\lambda) = 1 \pm 2hi - 4\beta h^2, \tag{9}$$

and for stability we must have

$$\begin{aligned} &|Q(h\lambda)|^2 = (1-4\beta h^2)^2 + 4h^2 \le 1 \Leftrightarrow 1 - 8\beta h^2 + 16\beta^2 h^4 + 4h^2 \le 1 \Leftrightarrow \\ &\Leftrightarrow -8\beta + 16\beta^2h^2 + 4 \le 0 \Leftrightarrow 4 - 8\beta + 16\beta^2 h^2 \le 0 \Leftrightarrow 16\beta^2h^2 \le 8\beta - 4. \end{aligned} \tag{10}$$

Hence it is necessary that $\beta > \frac{1}{2}$. Then, the following criterion for stability follows

$$h^2 \le \frac{2\beta - 1}{4\beta^2}. \tag{11}$$

(d) To compute the approximation with $\beta = \frac{1}{2}$, and $h = \frac{1}{2}$ we use the following steps:

$$\underline{w}_1^{*} = \underline{w}_0 + hf(t_0, w_0),$$

$$\underline{w}_1^{*} = \begin{pmatrix}1\\2\end{pmatrix} + \frac{1}{2}\left\{\begin{pmatrix}-1 & \frac{1}{2}\\ \frac{1}{2} & -1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}\right\} = \begin{pmatrix}1.5\\1.75\end{pmatrix}.$$

This can be used in the second step of the method:

$$\underline{w}_1 = \underline{w}_0 + (1-\beta)hf(t_0, \underline{w}_0) + \beta hf(t_1, \underline{w}_1^{*})$$

$$\underline{w}_1 = \begin{pmatrix}1\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\frac{1}{2}\\-\frac{1}{4}\end{pmatrix} + \frac{1}{2}\frac{1}{2}\left\{\begin{pmatrix}-1 & \frac{1}{2}\\ \frac{1}{2} & -1\end{pmatrix}\begin{pmatrix}1\frac{1}{2}\\1\frac{3}{4}\end{pmatrix} + \begin{pmatrix}1\\ \cos\frac{1}{2}\end{pmatrix}\right\} =$$

$$= \begin{pmatrix}\frac{43}{32}\\ 1\frac{5}{8} + \frac{1}{4}\cos\frac{1}{2}\end{pmatrix} = \begin{pmatrix}1.3438\\1.8444\end{pmatrix}.$$

2. (a) Taylor polynomials are:

$$\begin{aligned} f(0) &= f(0)\,, \\ f(h) &= f(0) + hf'(0) + \frac{h^2}{2}f''(0) + \frac{h^3}{6}f'''(\xi_1)\,, \\ f(2h) &= f(0) + 2hf'(0) + 2h^2 f''(0) + \frac{(2h)^3}{6}f'''(\xi_2)\,. \end{aligned}$$

We know that $Q(h) = \alpha_0 f(0) + \alpha_1 f(h) + \alpha_2 f(2h)$, which should be equal to $f'(0) + O(h^2)$. Since we lose one order by a first derivative we know that also the $O(h^2)$ should vanish. This leads to the following conditions:

$$\begin{array}{rcrcrcl} f(0): & \alpha_0 & + & \alpha_1 & + & \alpha_2 & = 0\,, \\ f'(0): & & & h\alpha_1 & + & h\alpha_2 & = 1\,, \\ f''(0): & & & \frac{h^2}{2}\alpha_1 & + & 2h^2\alpha_2 & = 0\,. \end{array}$$

(b) The truncation error follows from the Taylor polynomials:

$$f'(0) - Q(h) = f'(0) - \frac{-3f(0) + 4f(h) - f(2h)}{2h} = \frac{\frac{4h^3}{6}f'''(\xi_1) - \frac{8h^3}{6}f'''(\xi_2)}{2h},$$
$$= -\frac{1}{3}h^2 f'''(\xi).$$

(c) Note that

$$f'(0) - Q(h) = Kh^2 \tag{12}$$
$$f'(0) - Q(\frac{h}{2}) = K(\frac{h}{2})^2 \tag{13}$$

Subtraction gives:

$$Q(\frac{h}{2}) - Q(h) = Kh^2 - K\frac{h^2}{4} = 3K(\frac{h}{2})^2 \tag{14}$$

We choose $h = \frac{1}{2}$. Then $Q(h) = Q(\frac{1}{2}) = \frac{-3\times 0 + 4\times 0.4794 - 0.8415}{1} = 1.0761$ and $Q(\frac{h}{2}) = Q(\frac{1}{4}) = \frac{-3\times 0 + 4\times 0.2474 - 0.4794}{\frac{1}{2}} = 1.0204$. Combining (13) and (14) shows that

$$f'(0) - Q(\frac{1}{4}) = \frac{Q(\frac{1}{4}) - Q(\frac{1}{2})}{3} = -0.0186$$

(d) To estimate the rounding error we note that

$$|Q(h) - \hat{Q}(h)| = |\frac{-3(f(0) - \hat{f}(0)) + 4(f(h) - \hat{f}(h)) - (f(2h) - \hat{f}(2h))}{2h}|$$
$$\leq \frac{3|f(0) - \hat{f}(0)| + 4|f(h) - \hat{f}(h)| + |f(2h) - \hat{f}(2h)|}{2h} = \frac{8\epsilon}{2h} = \frac{4\epsilon}{h},$$

so $C_1 = 4$. Since only 4 digits are given the rounding error is: $\epsilon = 0.00005$.

(e) The total error is bounded by

$$|f'(0) - \hat{Q}(h)| = |f'(0) - Q(h) + Q(h) - \hat{Q}(h)|$$
$$\leq |f'(0) - Q(h)| + |Q(h) - \hat{Q}(h)|$$
$$\leq \frac{1}{3}h^2 + \frac{4\epsilon}{h} = g(h)$$

This is minimal if $g'(h) = 0$. Note that $g'(h) = \frac{2}{3}h - \frac{4\epsilon}{h^2}$. This implies that $h_{opt}^3 = 6 \cdot 0.00005$, so $h_{opt} = 0.0003^{\frac{1}{3}} = 0.0669$.