DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Tuesday 5 April 2005, 9:00-12:00

1. (a) Use the transformation:

$$\begin{aligned} y_1 &= \Phi \, , \\ y_2 &= \Phi' \, , \end{aligned}$$

This implies that

$$\begin{aligned} y_1' &= \Phi' = y_2 \, , \\ y_2' &= \Phi'' = -\Phi' - \frac{1}{2}\Phi = -y_2 - \frac{1}{2}y_1 = -\frac{1}{2}y_1 - y_2; \, , \end{aligned}$$

So the matrix A is given by $\begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}$.

(b) To compute the amplification factor one uses the test equation $y' = \lambda y$. Applying the Modified Euler method gives:

$$\begin{aligned} \text{predictor:} \quad \overline{w}_{n+1} &= w_n + hf(t_n, w_n), & (1) \\ \text{corrector:} \quad w_{n+1} &= w_n + \frac{h}{2}[f(t_n, w_n) + f(t_{n+1}, \overline{w}_{n+1})]. & (2) \end{aligned}$$

so

$$\begin{aligned} \text{predictor:} \quad \overline{w}_{n+1} &= w_n + h\lambda w_n, & (3) \\ \text{corrector:} \quad w_{n+1} &= w_n + \frac{h}{2}[\lambda w_n + \lambda(w_n + h\lambda w_n)]. & (4) \end{aligned}$$

Summarizing $w_{n+1} = (1 + h\lambda + \frac{1}{2}(h\lambda)^2)w_n$, which leads to the answer $Q(h\lambda) = 1 + h\lambda + \frac{1}{2}(h\lambda)^2$.

(c) The eigenvalues of the matrix A are $\lambda_1 = -\frac{1}{2} + \frac{i}{2}$ and $\lambda_2 = -\frac{1}{2} - \frac{i}{2}$. For stability it is needed that $|Q(h\lambda_1)| \leq 1$ and $|Q(h\lambda_2)| \leq 1$. Since $\lambda_2 = \bar{\lambda}_1$ it is sufficient to check the inequality $|Q(h\lambda_1)| \leq 1$. Using $h = 1$ we obtain $Q(h\lambda_1) = 1 + \lambda_1 + \frac{1}{2}\lambda_1^2 = \frac{1}{2} + \frac{i}{4}$. Note that $|Q(h\lambda_1)| = \sqrt{\frac{1}{4} + \frac{1}{16}} = 0.5590 \leq 1$, so the method is stable for $h = 1$.

(d) The local truncation error is defined by

$$\tau_{j+1} = \frac{y_{j+1} - z_{j+1}}{h}.$$

Using the testequation and the definition of z_{j+1} it appears that

$$z_{j+1} = Q(h\lambda)y_j.$$

For the exact solution we have:

$$y_{j+1} = e^{h\lambda}y_j.$$

This implies that

$$\tau_{j+1} = \frac{e^{h\lambda} - Q(h\lambda)}{h}y_j. \tag{5}$$

Note that

$$e^{h\lambda} = 1 + \lambda h + \frac{(\lambda h)^2}{2} + \mathcal{O}(h^3). \tag{6}$$

Furthermore by using the hint we can conclude that

$$\frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda} = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \mathcal{O}(h^3). \tag{7}$$

Combining (5), (6), and (7) we obtain that $\tau_{j+1} = \mathcal{O}(h^2)$.

(e) Again it is sufficient to check if $|Q(h\lambda_1)| \leq 1$. Using $\lambda_1 = -\frac{1}{2} + \frac{i}{2}$ it appears that

$$Q(h\lambda_1) = \frac{1 - \frac{h}{4} + \frac{hi}{4}}{1 + \frac{h}{4} - \frac{hi}{4}}$$

So

$$|Q(h\lambda_1)| = \sqrt{\frac{(1 - \frac{h}{4})^2 + (\frac{h}{4})^2}{(1 + \frac{h}{4})^2 + (\frac{h}{4})^2}} \leq 1.$$

The last inequality easily follows, because $h > 0$.

(f) The Jacobian is defined by:

$$\begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}$$

Using the definition it follows that

$$\begin{pmatrix} 0 & 1 \\ -\cos(y_1) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos\frac{\pi}{4} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{\sqrt{2}}{2} & 0 \end{pmatrix}$$

2. a After discretization by the use of finite differences one obtains

$$\frac{-w_{i-1} + 2w_i - w_{i-1}}{h^2} + x_i^2 w_i = x_i. \tag{8}$$

The truncation error is defined by

$$e_i = \frac{-y_{i-1} + 2y_i - y_{i+1}}{h^2} + x_i^2 y_i - x_i. \tag{9}$$

Taylor series of y_{i-1} and y_{i+1} around x_i, gives

$$\begin{aligned} y_{i+1} &= y_i + hy'(x_i) + \frac{h^2}{2!}y''(x_i) + \frac{h^3}{3!}y'''(x_i) + \frac{h^4}{4!}y''''(x_i) + O(h^5), \\ y_{i-1} &= y_i - hy'(x_i) + \frac{h^2}{2!}y''(x_i) - \frac{h^3}{3!}y'''(x_i) + \frac{h^4}{4!}y''''(x_i) - O(h^5), \end{aligned} \tag{10}$$

Substitution of the above expressions into the definition of the truncation error gives

$$\varepsilon_i = -y''(x_i) + O(h^2) + x_i^2 y(x_i) - x_i. \tag{11}$$

Using the differential equation $-y'' + x^2 y = x$ finally gives

$$\varepsilon_i = O(h^2). \tag{12}$$

b For this case we have $h = 0.25$, for the points $j \in \{1, 2, 3\}$, the discretization with $w_0 = 0$ and $w_4 = 1$:

$$\begin{aligned} &32w_1 - 16w_2 + \frac{1}{16}w_1 = \frac{1}{4}, \\ &-16w_1 + 32w_2 - 16w_3 + \frac{1}{4}w_2 = \frac{1}{2}, \\ &-16w_2 + 32w_3 + \frac{9}{16}w_3 = \frac{3}{4} + 16. \end{aligned} \tag{13}$$

Hence in matrix-vector form:

$$\begin{pmatrix} 32.0625 & -16 & 0 \\ -16 & 32.25 & -16 \\ 0 & -16 & 32.5625 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.5 \\ 16.75 \end{pmatrix} \tag{14}$$

c Since $h = \frac{1}{3}$, we have $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$ and $x_3 = 1$. Using linear interpolation, two adjacent gridpoints are taken into account. The minimum error is attained when the gridpoints x_1 and x_2 are used. The linear interpolation formula using points x_1 and x_2, gives:

$$P(0.5) = \frac{0.4444 + 0.7778}{2} = 0.6111. \tag{15}$$

The magnitude of the local truncation error is given by

$$\left|\frac{(x - x_1)(x - x_2)}{2}y''(\xi)\right| = \left|\frac{(0.5 - 1/3)(0.5 - 2/3)}{2} \cdot 1\right| = \frac{1}{72} = 0.0139. \tag{16}$$

d (i) The magnitude of the truncation error is given by

$$\left|\frac{y_2 - y_1}{h} - y'(x_1)\right| = \left|\frac{y(x_1) + hy'(x_1) + \frac{h^2}{2}y''(\xi) - y(x_1)}{h} - y'(x_1)\right| = \frac{h}{2}\left|y''(\xi)\right| = \frac{h}{2} = \frac{1}{6}. \tag{17}$$

(ii) The additional error is given by

$$\left|\frac{y_2 - y_1}{h} - \frac{w_2 - w_1}{h}\right| \leq \frac{2\varepsilon}{h} = \frac{2 \cdot 0.01}{\frac{1}{3}} = 0.06. \tag{18}$$