DELFT UNIVERSITY OF TECHNOLOGY
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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3 097 TU)
Tuesday 23 March 2004, 9:00-12:00

1. (a) Local truncation error $\tau_{j+1}(h)$:

$$\tau_{j+1}(h) := \frac{y_{j+1} - \overline{w}_{j+1}}{h} = \frac{y_{j+1} - y_j}{h} - (1-\beta) f(t_j, y_j) - \beta f(t_j, y_j + k_1) =$$

$$= y'(t_j) + \frac{h}{2} y''(t_j) + O(h_2) - (1-\beta) f(t_j, y_j) + \tag{1}$$

$$-\beta \left\{ f(t_j, y_j) + h f_t(t_j, y_j) + k_1 f_y(t_j, y_j) + O(h^2) \right\}.$$

Since $y' = f(t, y)$, use of the Chain Rule for differentiation gives

$$\begin{aligned} &y'(t_j) = f(t_j, y_j), \\ &y''(t_j) = f_t(t_j, y_j) + f_y(t_j, y_j) f(t_j, y_j), \\ &k_1 = h f(t_j, y_j) \text{ (by definition).} \end{aligned} \tag{2}$$

Hence after substitution into equation (1) we obtain:

$$\tau_{j+1}(h) = \frac{h}{2} \left\{ f_t(t_j, y_j) + f_y(t_j, y_j) f(t_j, y_j) \right\} - \beta h \left\{ f_t(t_j, y_j) + f_y(t_j, y_j) f(t_j, y_j) \right\} + O(h^2). \tag{3}$$

This implies that $\tau_{j+1}(h) = O(h^2)$ if $\beta = \frac{1}{2}$ and $\tau_{j+1}(h) = O(h)$ if $\beta \neq \frac{1}{2}$.

(b) We consider the amplification factor for the test-equation $y' = \lambda y$, then

$$k_1 = h\lambda w_j, \qquad k_2 = h\lambda(w_j + k_1) = (h\lambda + h^2\lambda^2) w_j. \tag{4}$$

Hence we have

$$w_{j+1} = w_j + (1-\beta) h\lambda w_j + \beta (h\lambda + h^2\lambda^2) w_j = \tag{5}$$

$$\tag{6}$$

$$= w_j \left\{ 1 + h\lambda + \beta h^2 \lambda^2 \right\} \Rightarrow Q(h\lambda) = 1 + h\lambda + \beta h^2 \lambda^2. \tag{7}$$

This $Q(h\lambda)$ is the amplification factor we need.

(c) Eigenvalues $\lambda_{1,2} = \pm i$. Hence the amplification factor is

$$Q(h\lambda) = 1 \pm hi - \beta h^2, \tag{8}$$

and for stability we must have

$$|Q(h\lambda)|^2 = (1-\beta h^2)^2 + h^2 \le 1 \Leftrightarrow 1 - 2\beta h^2 + \beta^2 h^4 + h^2 \le 1 \Leftrightarrow$$
$$\Leftrightarrow -2\beta + \beta^2 h^2 + 1 \le 0 \Leftrightarrow 1 - 2\beta + \beta^2 h^2 \le 0 \Leftrightarrow \beta^2 h^2 \le 2\beta - 1. \tag{9}$$

Hence it is necessary that $\beta > \frac{1}{2}$. Then, the following criterion for stability follows

$$h^2 \le \frac{2\beta - 1}{\beta^2}. \tag{10}$$

(d) (i) Euler Forward has a local truncation error of $O(h)$ (see Burden and Faires, page 266).

(ii) The amplification factor of Euler Forward is

$$Q(h\lambda) = 1 + h\lambda = 1 \pm hi \text{ (for our system)}. \tag{11}$$

Hence

$$|Q(h\lambda)|^2 = 1 + h^2 > 1 \text{ (for our system)}. \tag{12}$$

This implies that the Euler Forward method is always unstable for our system.

2. (a) The exact answer is 0.4. The composite Trapezoidal rule is given by

$$\frac{1}{2} \cdot \{\frac{1}{2} \cdot (-1)^4 + (-\frac{1}{2})^4 + 0^4 + (\frac{1}{2})^4 + \frac{1}{2} \cdot 1^4\} = \frac{9}{16} = 0.5625.$$

The difference with the exact answer is 0.1625.

(b) The rounding error is less than

$$h \cdot \{\frac{1}{2}\epsilon + \epsilon \ldots + \epsilon + \frac{1}{2}\epsilon\} \le n \cdot h \cdot \epsilon = (b-a) \cdot \epsilon.$$

(c) The Taylor polynomial is given by

$$P_1(x) = f(\frac{a+b}{2}) + (x - \frac{a+b}{2}) f'(\frac{a+b}{2})$$

whereas the truncation error is:

$$f(x) - P_1(x) = \frac{(x - \frac{a+b}{2})^2}{2} f''(\xi), \text{ with } \xi \in [a,b].$$

(d) Integrating this formula gives:

$$\int_a^b P_1(x)dx = \int_a^b f(\frac{a+b}{2}) + (x - \frac{a+b}{2}) f'(\frac{a+b}{2}) dx = (b-a) f\left(\frac{a+b}{2}\right).$$

Suppose that $M_2 = \max_{\xi \in [a,b]} |f''(\xi)|$. This implies that $|f(x) - P_1(x)| \leq \frac{(x-\frac{a+b}{2})^2}{2} M_2$. Integrating this formula gives:

$$|\int_a^b f(x)dx - (b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b |f(x) - P_1(x)|dx \leq \int_a^b \frac{(x-\frac{a+b}{2})^2}{2} M_2 = \frac{(b-a)^3}{24} M_2$$

(e) The composite rule is:

$$h \cdot \{f(a + \frac{1}{2}h) + f(a + \frac{3}{2}h) + \ldots + f(b - \frac{1}{2}h)\}$$

and the truncation error

$$\frac{n \cdot h^3}{24} \max_{\xi \in [a,b]} |f''(\xi)| = \frac{(b-a)h^2}{24} \max_{\xi \in [a,b]} |f''(\xi)|$$

(f) For the comparison we note that

- both methods have the same behavior with respect to rounding errors.
- the new method costs 1 function evaluation less than the Trapezoidal rule
- The truncation error of the new method is less than the truncation error of the Trapezoidal rule.

Conclusion: the new method is better than the Trapezoidal rule.