DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3 097 TU)
Wednesday 21 January 2004, 14:00-17:00

1. (a) Expand $y(1-h)$ in its Taylor series at t=1. Then, the truncation error (TE) is equal to $\frac{1}{2}hy''(\xi)$, with $\xi \in (1, 1-h)$.

   (b) Uniform acceleration may be assumed so the exact trajectory $y(t)$ is a quadratic function of t. This implies that y'' is constant and equal to 10, so TE$= 5h$. The measuring error in $y'(1)$: $\frac{0.2}{h}$. (Note: no measuring error in $y(1)$.)
   The total error is $5h + \frac{0.2}{h}$. Set the first derivative with respect to h equal to 0 $\rightarrow h_{opt} = 0.2$.

   (c) $y'(1) \approx \frac{y(1)-y(0.8)}{0.2} = \frac{-0.38}{0.2} = -19$ (m/sec). Maximal (absolute value of the) total error: $5h_{opt} + 0.2/h_{opt} = 5 * 0.2 + 0.2/0.2 = 2$ (m/sec).

   (d) Formulas using 3 or more points have third or higher derivatives in the expression for the TE. So, in this problem where the exact trajectory $y(t)$ may be assumed to be quadratic in t, these formulas have zero TE. Although the effect of measurement errors on $y'(1)$ will be larger than for the 2-point formula, there is a good chance that such formulas will give more accurate answers.

   (e) Use Taylor expansions

$$
\begin{array}{rcl}
3y(1) & = & 3y(1) \\
-4y(1-h) & = & -4[y(1) - hy'(1) + \frac{1}{2}h^2y''(1) - \frac{1}{6}h^3y'''(\xi_1)] \\
y(1-2h) & = & y(1) - 2hy'(1) + \frac{1}{2}(2h)^2y''(1) - \frac{1}{6}(2h)^3y'''(\xi_2) \\
--------- & = & ------------------------- \\
3y(1) - 4y(1-h) + y(1-2h) & = & 2hy'(1) + O(h^3),
\end{array}
$$

   and, dropping the $O(h^3)$-terms, the 3-point formula follows. Note that the quadratic terms cancel, such that the TE is equal to $[\frac{1}{3}y'''(\xi_1) - \frac{2}{3}y'''(\xi_2)]h^2$.

   (f) Total error = measuring error = $\frac{5*0.2}{2h}$ because in this application $y'''(t) = 0$. The total error is minimized by choosing h as large as possible, i.e. $h_{opt} = 0.5$. Then,

$$y'(1) \approx \frac{3y(1.0) - 4y(0.5) + y(0.0)}{2 * 0.5} = 3 * 0.0 - 4 * 8.8 + 14.9 = -20.3.$$

   Maximal error$= \frac{5*0.2}{2*0.5} = 1$ (m/sec); this is 50 percent smaller than the maximal error in (c).

(g) No, since the more points a difference formula uses the larger the (maximal) effect of measurement errors on $y'(1)$ (the TE remaining zero of course, as noted in (d)).

2. (a) To compute the eigenvalues one has to solve the following problem: determine λ such that $\det(A - \lambda I) = 0$. This leads to

$$\begin{vmatrix} -a-\lambda & -1 \\ 1 & -a-\lambda \end{vmatrix} = 0,$$

so $(-a-\lambda)^2 + 1 = 0$, $-a-\lambda = \pm i$, $\lambda_{1,2} = -a \pm i$.

(b) The amplification factor of Euler Forward is $Q(h\lambda) = 1 + h\lambda$. The method is stable if $|Q(h\lambda_{1,2})| \leq 1$. Substitution of λ_1 leads to:

$$|1 + h(-a+i)| \leq 1,$$

$$\sqrt{(1-ha)^2 + h^2} \leq 1,$$

$$1 - 2ha + h^2a^2 + h^2 \leq 1,$$

$$h^2(a^2+1) \leq 2ha,$$

$$h \leq \frac{2a}{a^2+1}.$$

For λ_2 the same inequalities hold.

(c) The test equation is $y' = \lambda y$. The amplification factor is defined such that $w_{j+1} = Q(h\lambda)w_j$. This leads to:

$$w_{j+1} = w_j + \frac{h}{2}(\lambda w_j + \lambda w_{j+1}),$$

$$(1 - \frac{h}{2}\lambda)w_{j+1} = (1 + \frac{h}{2}\lambda)w_j,$$

$$w_{j+1} = \frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda}w_j,$$

so $Q(h\lambda) = \frac{1+\frac{h}{2}\lambda}{1-\frac{h}{2}\lambda}$.

(d) We first check that $|Q(h\lambda_1)| \leq 1$.

$$|Q(h\lambda_1| = \frac{|1 + \frac{h}{2}\lambda|}{|1 - \frac{h}{2}\lambda|} = \frac{\sqrt{(1 - \frac{h}{2}a)^2 + \frac{h^2}{4}}}{\sqrt{(1 + \frac{h}{2}a)^2 + \frac{h^2}{4}}} \leq 1.$$

The last inequality holds because $(1 - \frac{h}{2}a)^2 < (1 + \frac{h}{2}a)^2$.

(e) Note that $\boldsymbol{w}_0 = \boldsymbol{u}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So one step of Euler Forward gives:

$$\begin{aligned} \boldsymbol{w}_1 &= \boldsymbol{w}_0 + h[A\boldsymbol{w}_0 + \boldsymbol{g}(0)] \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}. \end{aligned}$$

(f) One step of the trapezoidal rule gives:

$$\begin{aligned} \boldsymbol{w}_1 &= \boldsymbol{w}_0 + \frac{h}{2}[A\boldsymbol{w}_0 + \boldsymbol{g}(0) + A\boldsymbol{w}_1 + \boldsymbol{g}(h)] \\ \boldsymbol{w}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}\boldsymbol{w}_1 + \frac{1}{4}\begin{pmatrix} \frac{1}{2} \\ 1 + \frac{1}{2} \end{pmatrix}. \end{aligned}$$

Bringing the unknown vector to the left-hand side leads to

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix}\right]\boldsymbol{w}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{3}{4} \\ -\frac{1}{4} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{8} \\ \frac{3}{8} \end{pmatrix}.$$

This can be simplified into

$$\begin{pmatrix} \frac{3}{2} & \frac{1}{4} \\ -\frac{1}{4} & \frac{3}{2} \end{pmatrix}\boldsymbol{w}_1 = \begin{pmatrix} \frac{3}{8} \\ 1\frac{3}{8} \end{pmatrix}.$$

After Gaussian elimination one obtains:

$$\boldsymbol{w}_1 = \begin{pmatrix} 0.0946 \\ 0.9324 \end{pmatrix}.$$