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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Friday 27 August 2004, 9:00-12:00

1. (a) The method is explicit, because the value w_{k+1} is only present on the left-hand side.

   (b) The local truncation error $\tau_{j+1}(h)$ is given by

$$\tau_{j+1}(h) = \frac{y_{k+1} - \overline{w}_{k+1}}{h}, \tag{1}$$

where

$$\overline{w}_{k+1} = y_k + hf(t_k + \frac{1}{2}h, y_k + \frac{1}{2}hf(t_k, y_k)). \tag{2}$$

We develop y_{k+1} in a Taylor polynomial:

$$y_{k+1} = y_k + hy'_k + \frac{h^2}{2!}y''_k + O(h^3). \tag{3}$$

Using the Taylor polynomial in two variables one obtains

$$f(t_k+\frac{1}{2}h, y_k+\frac{1}{2}hf(t_k, y_k)) = f(t_k, y_k)+\tfrac{1}{2}h\tfrac{\partial f}{\partial t}(t_k, y_k)+\tfrac{1}{2}h\tfrac{\partial f}{\partial y}(t_k, y_k)f(t_k, y_k)+O(h^2) \tag{4}$$

Note that y is a solution of the differential equation $y' = f(t, y)$. This implies: $f(t_k, y_k) = y'_k$. Differentiation of the differential equation yields

$$y'' = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}f \tag{5}$$

Combination of (4) and (5) shows:

$$f(t_k + \frac{1}{2}h, y_k + \frac{1}{2}hf(t_k, y_k)) = y'_k + \tfrac{1}{2}hy''_k + O(h^2). \tag{6}$$

Substituting (2), (3), and (6) in (1) leads to the required result.

   (c) The method has an error of $O(h^2)$, so

$$y(1) - w_1(1) \approx Ch_1^2 \tag{7}$$

and

$$y(1) - w_2(1) \approx Ch_2^2 = \tfrac{1}{4}Ch_1^2 \tag{8}$$

Subtract (8) from (7),which yields:

$$w_2(1) - w_1(1) = \tfrac{3}{4}Ch_1^2$$

Combined with (8) one has:

$$y(1) - w_2(1) \approx \tfrac{1}{3}(w_2(1) - w_1(1))$$

which implies that $K = \frac{1}{3}$.

(d) Apply the method to the test equation $y' = \lambda y$. This yields:

$$w_{k+1} = w_k + h\lambda(w_k + \tfrac{1}{2}h\lambda w_k)$$

so

$$w_{k+1} = (1 + h\lambda + \frac{(h\lambda)^2}{2})w_k$$

(e) We first rewrite the second order differential equation into a system of first order equations. Therefore we define $u_1 = y$ and $u_2 = y'$. The resulting linear system is:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}.$$

The eigenvalues of the matrix $\begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}$ are given by $\lambda_{1,2} = -1 \pm 2i$, so $h\lambda = -0.1 \pm 0.2i$ for $h = 0.1$. Substituting this value into the amplification factor gives

$$Q(-0.1 + 0.2i) = 1 - 0.1 + 0.2i + \frac{(-0.1 + 0.2i)^2}{2}.$$

Working this out gives:

$$Q(-0.1 + 0.2i) = 0.885 + 0.18i$$

and $|Q(-0.1 + 0.2i)| = 0.9031 < 1$. So the integration can be done in a stable way for this step size.

2. (a) After discretization one obtains:

$$\frac{-u_{j-1} + 2u_j - u_{j+1}}{h^2} = f_j.$$

The truncation error is defined as:

$$\epsilon_j = \frac{-y_{j-1} + 2y_j - y_{j+1}}{h^2} - f_j. \tag{9}$$

Taylor expansion of y_{j+1} and y_{j-1} gives:

$$y_{j+1} = y_j + hy'_j + \frac{h^2}{2!}y''_j + \frac{h^3}{3!}y'''_j + O(h^4)$$

$$y_{j-1} = y_j - hy'_j + \frac{h^2}{2!}y''_j - \frac{h^3}{3!}y'''_j + O(h^4)$$

Substituting this into (9) and using $-y'' = f$ shows that the truncation error is of order $O(h^2)$.

(b) We replace the left boundary condition by

$$\frac{u_1 - u_{-1}}{2h} = 0$$

The truncation error is defined as:

$$\epsilon = \frac{y_1 - y_{-1}}{2h} - y'_0 \tag{10}$$

Using the Taylor expansion

$$y_1 = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + O(h^3)$$

$$y_{-1} = y_0 - hy'_0 + \frac{h^2}{2!}y''_0 + O(h^3)$$

shows that $\epsilon = O(h^2)$.

(c) In this case $h = 0.25$. For the indices $j = 0,1,2$ and 3 we can write out the equations as given in (a). Using the boundary conditions: $u_4 = 1$ and $u_{-1} = u_1$ leads to the resulting system:

$$\begin{pmatrix} 32 & -32 & 0 & 0 \\ -16 & 32 & -16 & 0 \\ 0 & -16 & 32 & -16 \\ 0 & 0 & -16 & 32 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.25 \\ 0.5 \\ 16.75 \end{pmatrix}.$$

(d) In this case $n = 3$ and $h = \frac{1}{3}$. The grid points are $x_0 = 0,\ x_1 = \frac{1}{3},\ x_2 = \frac{2}{3}$ and $x_3 = 1$. With linear interpolation only two grid points can be used. The smallest error occurs when we use the grid points x_1 and x_2. Substituting the data into the formula for linear interpolation yields: $p(0.5) = (0.4444 + 0.7778)/2 = 0.6111$. The truncation error is:

$$\frac{(x - x_1)(x - x_2)}{2}y''(\xi)$$

Using $x = 0.5,\ x_1 = \frac{1}{3},\ x_2 = \frac{2}{3}$ and y" = -1 (y satisfies the differential equation) shows that the absolute value of the truncation error is less than or equal to: $\frac{1}{72} = 0.0139$.

(e) The absolute value of the extra error is now less than or equal to:

$$\frac{|y_1 - u_1| + |y_2 - u_2|}{2} \leq 0.01$$

This estimate is comparable to the occurrence of a rounding error in the value of y.