Delft University of Technology

Faculty of Civil Engineering and Geosciences

TU Delft

AESB2122: Signals and Systems with Python (5 EC)
Final Exam

1st November, 2021

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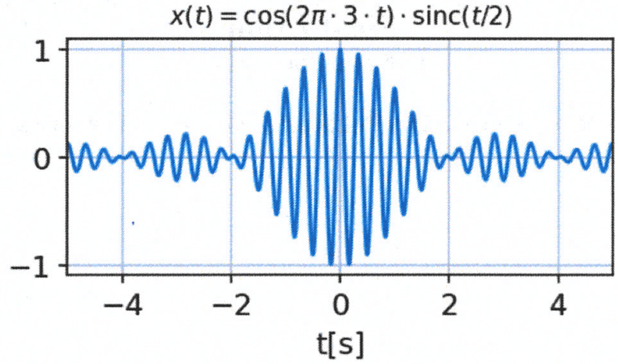
1. This test consists of 3 open questions, adding up to 50 points. .
2. This exam counts for 50% of the final grade of the course. The theory past has a relative weight of 60%.
3. The minimum grade to pass this course is 5.75, which will be rounded to 6.0.
4. The maximum grade for this course is 10.0.
5. This is an **open-book**, written test: book, slide print-outs, or any printed material is allowed. **Some useful formulas are included in the last page.**
6. You can use the computer, gather information from internet, but you cannot ask for help or collaborate from/with anyone.
7. Answers can be given in English or Dutch.
8. By default, time *t* is expressed in seconds [s], and frequency *f* in Hertz [Hz].

Question 1 [25pt]

Let us consider the following signal, which we can call a *wave packet*.

$$x(t) = \cos(2\pi f_0 t) \cdot \mathrm{sinc}\left(\frac{t}{T_p}\right)$$

where we also define $f_p = 1/T_p$, and we can assume that f_0 is much larger than f_p. For example, the figure below illustrates the case $f_0 = 3$, $T_p = 2$.



a. **[6pt]** Using **exclusively** the Fourier-transform pairs and properties **listed at the end of this exam**, show that its Fourier transform is given by

$$X(f) = \frac{T_p}{2}\left(\Pi\left(\frac{f+f_0}{f_p}\right) + \Pi\left(\frac{f-f_0}{f_p}\right)\right),$$

where $f_p = 1/T_p$.

b. **[3pt]** Determine energy of $x(t)$, E_x.

Let us assume that we filter this signal with impulse response

$$h(t) = \cos(2\pi f_0 t) \cdot \mathrm{sinc}\left(\frac{t}{2 \cdot T_p}\right)$$

obtaining the signal

$$y_1(t) = x(t) * h(t)$$

c. **[1pt]** What kind of filter is this?

d. **[4pt]** Determine $y_1(t)$.

e. **[1pt]** Determine the energy of $y_1(t)$.

We are often interested in the square of a signal, so let us consider now

$$x_2(t) = \left(x(t)\right)^2$$

f. **[3pt]** Determine its Fourier transform.

Now let us consider a superposition of two wave packets with slightly different frequencies:

$$x_s(t) = \cos(2\pi f_1 t) \cdot \mathrm{sinc}\left(\frac{t}{T_p}\right) + \cos(2\pi f_2 t) \cdot \mathrm{sinc}\left(\frac{t}{T_p}\right)$$

g. **[3pt]** Sketch its Fourier transform, $X_s(f)$, for the following two cases

- $\Delta f = f_2 - f_1 = \frac{2}{T_p}$
- $\Delta f = f_2 - f_1 = \frac{1}{2T_p}$

h. **[2pt]** What is the minimum Δf for which we can (easily) identify the frequencies of the two wave packets?

i. **[2pt]** Imagine that we would do this analysis by sampling the signals at a Nyquist compliant frequency and performing a DFT on the sampled data. We have learned in theory that the frequency resolution of the DFT is given by $\frac{F_s}{N_s} = \frac{1}{T_{\mathrm{meas}}}$. Does this mean that we would be able to discriminate the two wave packets by making $T_{\mathrm{meas}} > \frac{1}{\Delta f}$ no matter how small we make Δf? Please (think and) motivate your answer.

Question 2 [13pt]
In many applications (for example in radar remote sensing, radioastronomy, or for the detection of gravitational waves) we use interferometric techniques, involving the multiplication of two signals. Let us consider the two signals $x_1(t)$ and $x_2(t)$, with their corresponding Fourier transforms given by

$$X_1(f) = \Pi\left[\frac{f}{W}\right]$$

and

$$X_2(f) = e^{-j2\pi f\cdot\Delta t}\cdot\Pi\left[\frac{f}{W}\right]$$

with W=2 MHz.

a. **[2pt]** Determine the time-domain expression for both signals, i.e. $x_1(t)$ and $x_2(t)$.
b. **[2pt]** What does the frequency-domain factor $e^{-j2\pi f\cdot\Delta t}$ imply in the time-domain?
c. **[2pt]** Assuming that we sample each signal separately, what would be the minimum sampling frequency required to avoid loss of information.

Let us consider now the product of these two signals

$$y(t) = x_1(t)\cdot x_2(t)$$

Finding the Fourier transform of this signal would be a nice homework exercise. However, let us keep things simple by assuming $\Delta t = 0$ (in other words, let us assume $y(t) = x_1^2(t)$). In this case we have

$$Y(f) = W\cdot\Lambda\left(\frac{f}{W}\right)$$

and

$$y(t) = W^2\cdot\mathrm{sinc}^2(W\cdot t)$$

Let us assume that we sample $x_1(t)$ and $y(t)$ with a sampling frequency $F_s = 2$ MHz.

d. **[3pt]** Represent, in the frequency domain, the ideally sampled signals.
e. **[2pt]** Sketch the two discrete signals, i.e. $x_1[n]$ and y[n] for values of n between -3 and 3 (or more).
f. **[2pt]** Is 2 MHz a high enough sampling frequency to allow a perfect reconstruction of y(t)? If not, what is the minimum required frequency in this case?

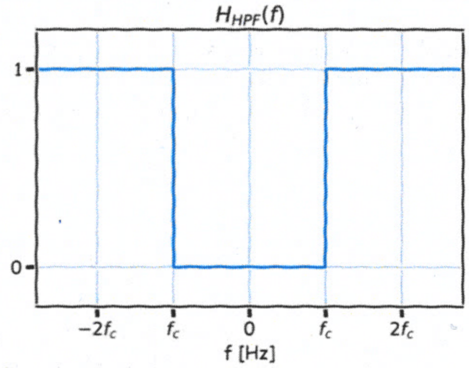
Question 3 [12pts] Let us consider the following signals:

$$x(t) = \cos(2\pi \cdot 4 \cdot t) + \cos(2\pi \cdot 14 \cdot t) - \cos(2\pi \cdot 24 \cdot t)$$

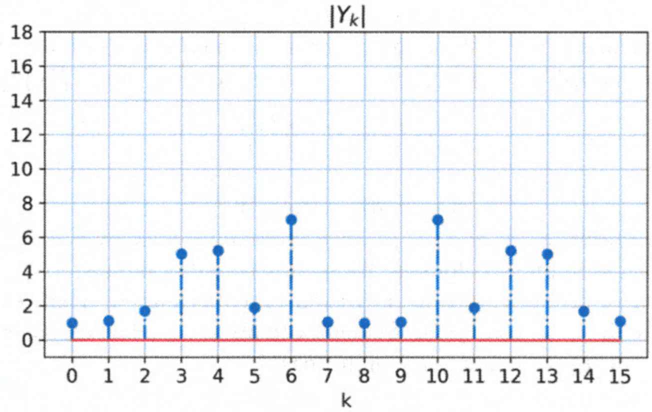
and

$$y(t) = h_{HPF}(t) * x(t)$$

where $h_{HPF}(t)$ is the impulse response of an ideal high-pass filter with the frequency response sketched below:



We are going to sample these signals taking 16 samples at a sampling frequency F_s, and then look at the values of the DFT. For example, if $F_s = 64$, and $f_c = 10$ Hz the figure below represents the amplitude of samples of the DFT of $y[n]$, Y_k.



a. **[6pt]** Explain what you see in the figure addressing the following:
   - Which of the three components of the signal do we see?
   - To which values of k correspond the different frequency components present in y(t)?
   - How come all Y_k are non-zero if we only have two or three frequencies in our signal?

b. **[3pt]** $|X_k|$ for $F_s = 64$ (as before)

c. **[3pt]** $|Y_k|$ for $F_s = 32$ and for a cut-off frequency $f_c = 100$ Hz.

Some Fourier transform pairs

$\mathcal{F}\left\{\Pi\left(\frac{t}{T}\right)\right\} = T \cdot \text{sinc}(T \cdot f)$	$\mathcal{F}\{2W \text{sinc}^2(2Wt)\} = \Lambda\left(\frac{f}{2W}\right)$
$\mathcal{F}\{\delta(t - t_0)\} = e^{-j2\pi t_0 f}$	$\mathcal{F}\{e^{j2\pi f_0 t}\} = \delta(f - f_0)$
$\mathcal{F}\{u(t) \cdot e^{-\alpha t}\} = \frac{1}{\alpha + j2\pi f}$	$\mathcal{F}\{\cos(2\pi f_0 t)\} = \frac{1}{2}\delta(f + f_0) + \frac{1}{2}\delta(f - f_0)$

Some properties of the Fourier transform

Linearity	$\mathcal{F}\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 X_2(f) + a_2 X_2(f)$
Time scaling	$\mathcal{F}\{x(a \cdot t)\} = \frac{1}{\lvert a \rvert} \cdot X\left(\frac{f}{a}\right)$
Time delay	$\mathcal{F}\{x(t - t_0)\} = X(f) \cdot e^{-j2\pi t_0 f}$
Duality	Given $\mathcal{F}\{x(t)\} = X(f)$, $\mathcal{F}\{X(t)\} = x(-f)$
Convolution	$\mathcal{F}\{x(t) * y(t)\} = X(f) \cdot Y(f)$
Multiplication	$\mathcal{F}\{x(t) \cdot y(t)\} = X(f) * Y(f)$

Energy:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$