Method of Characteristics

In Boyce & DiPrima only some (very special) second order partial differential equations are considered. Also first order PDE's do pop up, and we will consider some examples of linear equations.

Simplest example $\dfrac{\partial w}{\partial t} + c \cdot \dfrac{\partial w}{\partial x} = 0$

Question: find all solutions $w = w(x,t)$ of this equation.

Idea: consider the rate of change of w following a 'moving observer', i.e. $x = x(t)$.

$$\text{applying the chain rule:} \quad \frac{d}{dt}\Big(w(x(t),t)\Big) = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial t}$$

If the 'observer' moves with a constant velocity c: $\dfrac{dx}{dt} = c$.

then it follows from $\dfrac{\partial w}{\partial t} + c\dfrac{\partial w}{\partial x} = 0$ that $\dfrac{d}{dt}\Big(w(x(t),t)\Big) = 0$, so w is constant for an observer that moves with a velocity c.

Put otherwise:
w is **constant** along the curves (in this cases straight lines) $x - ct = x_0$.

These curves are called the **characteristics** of the solution.

The 'general solution' is then given by $w(x,t) = P(x - ct)$, where P is an arbitrary (differentiable) function of one variable.

If w is given on a curve that intersects each characteristic in one point, then $w(x,t)$ is uniquely determined for any point (x,t).
Usually this curve is the line $t = 0$, in which case one speaks of an **initial condition**. Specifically, if w has to satisfy $w(x,0) = g(x)$
then P is specified by the constraint $w(x,0) = P(x - c \cdot 0) = P(x) = g(x)$, and the solution becomes $w = g(x - ct)$.

Another way to view this: the characteristic going through the 'general' point (x,t) intersects the axis $t = 0$ in the point $(x - ct, 0)$, so $w(x,t) = w(x - ct, 0) = g(x - ct)$

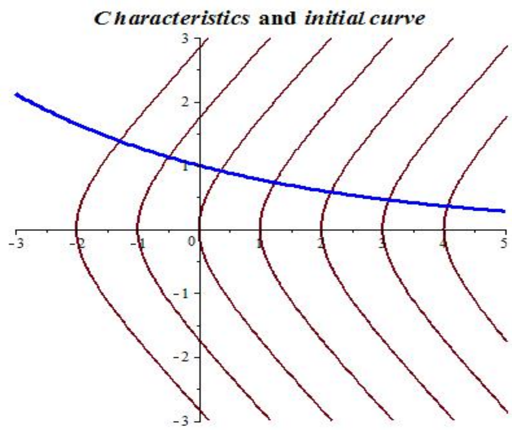
Slightly more general $\dfrac{\partial w}{\partial t} + c(x,t) \cdot \dfrac{\partial w}{\partial x} = 0$

Again we follow the solution $w = w(x,t)$ along curves $x = x(t)$ satisfying $\dfrac{dx}{dt} = c(x,t)$, which like before are called the **characteristic curves**.
Along these characteristic curves the solutions $w = w(x,t) = w(x(t),t)$ are constant.

Often the description of the caracteristics can only be given implicitly, say as $\kappa(x,t) = C$ in that case $w(x,t) = P(\kappa(x,t))$ satisfies the PDE for any (differentiable) function P.

The solution of the initial value problem $\begin{cases} \dfrac{\partial w}{\partial t} + c(x,t) \cdot \dfrac{\partial w}{\partial x} = 0 \\ w(x,0) = g(x) \end{cases}$

can be found explicitly only if it is possible to find the intersection $(x_0, 0)$ of the characteristic curve through the point (x,t) and the line $t = 0$.



Example 1 $x\dfrac{\partial w}{\partial t} + t\,\dfrac{\partial w}{\partial x} = 0,\ w(x,0) = g(x),$ for $x > 0$.

The PDE can be rewritten as $\dfrac{\partial w}{\partial t} + \dfrac{t}{x}\dfrac{\partial w}{\partial x} = 0.$

For the characteristics we have to solve $\dfrac{dx}{dt} = \dfrac{t}{x}$, which is a separable differential equation:

$$\frac{dx}{dt} = \frac{t}{x} \ \Rightarrow\ x\,dx = t\,dt \ \Rightarrow\ \int x\,dx = \int t\,dt \ \Rightarrow\ \frac{1}{2}x^2 = \frac{1}{2}t^2 + C$$

The implicit form of the characteristics: $x^2 - t^2 = K$.

This gives the general solution $w(x,t) = P(x^2 - t^2)$.

To satisfy the intial condition: $w(x,0) = P(x^2) = g(x)$, so $P(x) = g(\sqrt{x})$.

It can be easily checked that $w(x,t) = g(\sqrt{x^2 - t^2})$ indeed satisfies both PDE and initial condition.

Example 2 $(3x^2 + 3)\dfrac{\partial w}{\partial t} + 6t\,\dfrac{\partial w}{\partial x} = 0.$

Again the characteristics are the solutions of a separable DE:

$$\frac{dx}{dt} = \frac{6t}{3x^2 + 3} \ \Rightarrow\ (3x^2 + 3)\,dx = 6t\,dt \ \Rightarrow\ x^3 + 3x = 3t^2 + C$$

So the characteristics are implicitly given by $\kappa(x,t) = x^3 + 3x - 3t^2 = C$, and the 'general solution' becomes $w(x,t) = P(x^3 + 3x - 3t^2 - c)$.
In this case it is not possible to find explicitly the solution that satisfies the initial condition $w(x,0) = g(x)$.

Applications

A first order PDE often pops up as a **continuity equation**

Suppose some density (e.g. water, pollution, cars) is transported over a line (pipe, canal, highway). Let $\rho = \rho(x,t)$ be the density at position x at time t and $v = v(x,t)$ the velocity. Sometimes the 'flow' $q = q(x,t)$, i.e. the 'mass' transported per time unit is introduced: $q(x,t) = \rho(x,t)v(x,t)$.
The total mass in some interval $a \leq x \leq b$ at some specific time t equals

$$m = \int_a^b \rho(x,t)\,dx$$

Likewise the inflow at a minus the outflow at b during a small interval Δt equals

$$m_{in} - m_{out} = \big(q(a,t) - q(b,t)\big)\ \Delta t$$

If there are no 'sources' and no 'sinks' in the interval (i.e. no mass is created or annihilated) then the change of mass dm/dt is only due to this in- and outflow, which gives

$$\frac{dm}{dt} = \frac{d}{dt}\left[\int_a^b \rho(x,t)\,dx\right] = q(a,t) - q(b,t)$$

which can be rewritten as

$$\int_a^b \frac{\partial \rho}{\partial t}(x,t)\,dx = q(a,t) - q(b,t) = -\int_a^b \frac{\partial q}{\partial x}(x,t)\,dx$$

Since this equality holds on any small interval it may be concluded that

$$\frac{\partial \rho}{\partial t} = -\frac{\partial q}{\partial x} = -\frac{\partial(\rho v)}{\partial x} = -\rho\,\frac{\partial v}{\partial x} - v\,\frac{\partial \rho}{\partial x} \qquad (*)$$

If some further assumptions are made with respect to the relation between flow and densities, like $q = g(\rho)$, a PDE for ρ results:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial g(\rho)}{\partial x} = -g'(\rho)\frac{\partial \rho}{\partial x} \implies \frac{\partial \rho}{\partial t} + g'(\rho)\frac{\partial \rho}{\partial x} = 0$$

This equation is equivalent to the PDE we considered above.

In the theory of flow equations, I came across equation $(*)$ disguised as

$$-\rho_w \frac{\partial q_x}{\partial x} - q_x \frac{\partial \rho_w}{\partial x} = \rho_w S_s \frac{\partial h}{\partial t}$$

where ρ_w is the density of water, q_x the flow in the x-direction, S_s the 'specific storage', and h the 'head'.
If it is further assumed that $\rho_w \dfrac{\partial q_x}{\partial x} \gg q_x \dfrac{\partial \rho_w}{\partial x}$, which often seems to be the case, the equation reduces to

$$-\frac{\partial q_x}{\partial x} = S_s \frac{\partial h}{\partial t}$$

Plugging in Darcy's law $\left(\dfrac{\partial q_x}{\partial x} = -K_x \dfrac{\partial h}{\partial x} \right)$ yields a second order PDE for the head:

$$\frac{\partial}{\partial x} \left[K_x \frac{\partial h}{\partial x} \right] = S_s \frac{\partial h}{\partial t}$$

where K_x denotes the 'conductivity' in the x-direction.
This equation generalizes to two and three dimensions:

$$\frac{\partial}{\partial x} \left(K_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(K_y \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial y} \left(K_z \frac{\partial h}{\partial z} \right) = S_s \frac{\partial h}{\partial t}$$

which in the case of homogeneous (K_x, K_y, K_z are constant) anisotropic ($K_x = K_y = K_z$) flow simplifies to

$$K \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \right) = K \, \nabla^2 h = S_s \frac{\partial h}{\partial t}$$

Concluding remarks

In fact the second 'slightly general' example could be called a linear, homogeneous first order PDE, the general linear first order PDE would then be

$$a(x,t) \frac{\partial w}{\partial t} + b(x,t) \cdot \frac{\partial w}{\partial x} = c(x,t)$$

Like in the case of linear ODE's it is not difficult to show that the general solution can be split into a homogeneous and a 'particular' part: $w(x,t) = w_H(x,t) + w_P(x,t)$.

The method of characteristics is also more or less applicable for the so-called *quasi-linear* PDE

$$a(x,t,w) \frac{\partial w}{\partial t} + b(x,t,w) \cdot \frac{\partial w}{\partial x} = c(x,t,w)$$

In both cases explicit solutions can seldom be found.
We will not pursue these extensions further.

Exercises Method of Characteristics

1. Solve the PDE $(1+t^2)\dfrac{\partial u}{\partial t}+2xt\dfrac{\partial u}{\partial x}=0$ with the BC $u(x,0)=b(x)$.

2. Solve the PDE $\dfrac{\partial u}{\partial t}+e^{-x}\,\dfrac{\partial u}{\partial x}=0$ with the BC $u(x,0)=b(x)$.

3. Solve the PDE $u_t+2xt\,u_x=0$ with the BC $u(x,0)=b(x)$.

4. Show that the solution of the general first order linear PDE

$$a(x,t)\frac{\partial w}{\partial t}+b(x,t)\cdot\frac{\partial w}{\partial x}=c(x,t)$$

   can be written as $w(x,t)=w_H(x,t)+w_P(x,t)$, where $w_H(x,t)$ satisfies the homogeneous equation

$$a(x,t)\frac{\partial w}{\partial t}+b(x,t)\cdot\frac{\partial w}{\partial x}=0$$

5. Use the observation of the above exercise to solve the boundary value problem

$$x\,\frac{\partial u}{\partial x}+y\,\frac{\partial u}{\partial y}=x,\quad u(1,y)=y^2$$

   (A particular solution is easy to 'guess'.)

6. Find the general solution of the linear PDV

$$2\,\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=g(x)$$

7. Find the general solution of the linear PDV

$$2\,\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=g(x+y)$$

   Hint: the substitution $\begin{cases} s=x \\ t=x+y \end{cases}$ will bring the equation in the form of the previous exercise.

Solutions to the exercises

1. Solve the PDE $(1+t^2)\dfrac{\partial u}{\partial t} + 2xt\dfrac{\partial u}{\partial x} = 0$ with the BC $u(x,0) = b(x)$.

   Solution:
   The characteristices follow from $(1+t^2)\,dx = 2xt\,dt$, which is separable:

   $$\frac{dx}{x} = \frac{2t\,dt}{1+t^2} \Leftrightarrow \int \frac{dx}{x} = \int \frac{2t\,dt}{1+t^2} \Leftrightarrow \ln|x| = \ln(1+t^2) + K \Leftrightarrow \frac{x}{1+t^2} = c$$

   The characteristics are given by $\dfrac{x}{1+t^2} = c$, so the general solution becomes $u(x,t) = P(x/(1+t^2))$.

   To satisfy the boundary condition: $u(x,0) = P(x/(1+0^2) = P(x) = b(x)$, which gives the solution $u(x,t) = b\left(\dfrac{x}{1+t^2}\right)$

2. Solve the PDE $\dfrac{\partial u}{\partial t} + e^{-x}\dfrac{\partial u}{\partial x} = 0$ with the BC $u(x,0) = b(x)$.

   Solution:
   The characteristices follow from $dx = e^{-x}\,dt$, which is separable:

   $$e^x\,dx = dt \Leftrightarrow \int e^x\,dx = \int dt \Leftrightarrow e^x = t + K \Leftrightarrow e^x - t = K$$

   The characteristics are given by $e^x - t = c$, so the general solution becomes $u(x,t) = P(e^x - t)$.

   To satisfy the boundary condition: $u(x,0) = P(e^x - 0) = P(e^x) = b(x)$. So $P(x) = P(e^{\ln x}) = b(\ln x)$, which gives the solution $u(x,t) = P(e^x - t) = b(\ln(e^x - t))$.
   It is readily checked that this $u(x,t)$ indeed satisfies both PDE and BC.

3. Solve the PDE $u_t + 2xt\,u_x = 0$ with the BC $u(x,0) = b(x)$.
   Solution:
   The characteristices follow from $dx = 2xt\,dt$, which is separable:

   $$\frac{dx}{x} = 2t\,dt \Leftrightarrow \ln|x| = t^2 + C \Leftrightarrow |x| = e^{t^2+C} = Ke^{t^2} \Leftrightarrow xe^{-t^2} = K$$

   The characteristics are given by $xe^{-t^2} = K$, so the general solution becomes $u(t,x) = P(xe^{-t^2})$, for an arbitrary function P.

   To satisfy the boundary condition: $u(0,x) = P(xe^0) = P(x) = b(x)$. So $P(x) = b(x)$, which gives the solution $u(t,x) = P(xe^{-t^2}) = b(xe^{-t^2})$.

OR: on the characteristics $u(t,x) = u(t, Ke^{t^2}) = c$,
so to satisfy $u(0,x) = b(x)$: $u(0, Ke^0) = u(0,K) = b(K)$.
So $u(t, Ke^{t^2}) = b(K)$, and in a 'general' point (t,x):
the solution becomes $u(x,t) = u(t, (xe^{-t^2})e^{t^2}) = b(xe^{t^2})$.

4. Show that the solution of the general first order linear PDE

$$a(x,t)\frac{\partial w}{\partial t} + b(x,t) \cdot \frac{\partial w}{\partial x} = c(x,t)$$

can be written as $w(x,t) = w_H(x,t) + w_P(x,t)$, where $w_H(x,t)$ satisfies the homogeneous equation

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$$x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = x, \quad u(1,y) = y^2$$

(A particular solution is easy to 'guess'.)

6. Find the general solution of the linear PDV

$$2\,\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = g(x)$$

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