1 (a) oplossen $\begin{cases} y'' + y = g(t) \\ y(0)=0,\ y'(0)=1 \end{cases}$, $g(t) = u_0(t) - u_{2\pi}(t)$
$= 1 - u_{2\pi}(t)$
(want bekijken alleen $t \geq 0$)

Laplace transformeren:

$$s^2 Y(s) - 1 + Y(s) = \frac{1}{s} - \frac{e^{-2\pi s}}{s}$$

$$\Rightarrow Y(s) = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{1}{s(s^2+1)} \cdot e^{-2\pi s}$$

Breuksplitsen: $\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \xrightarrow{\text{rekenen}} = \frac{1}{s} - \frac{s}{s^2+1}$

Terugtransformeren, met gebruik van tabel:

$$y(t) = \sin t + [1 - \cos t] - [1 - \cos(t-2\pi)] \cdot u_{2\pi}(t)$$

$$= \begin{cases} \sin t + 1 - \cos t, & 0 \leq t \leq 2\pi \\ \sin t, & 2\pi < t \end{cases}$$

(b) nu $s^2 Y(s) - 1 + Y(s) = G(s)$

$$\Rightarrow Y(s) = \frac{1}{s^2+1} + \frac{1}{s^2+1} \cdot G(s)$$

$$\Rightarrow y(t) = \sin t + (\sin * g)(t)$$

$$= \sin t + \int_0^t g(u) \cdot \sin(t-u)\, du$$

(c) Vanwege $\sin\alpha \cdot \sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$

geldt $\int_0^t \sin u \cdot \sin(t-u)\, du = \frac{1}{2} \int_0^t \cos(\overset{2u-t}{\cdot}) - \cos t\, du = \dots =$

$= +\frac{1}{2} \sin t - \frac{1}{2} t \cos t$. <u>Antwoord</u>: $y(t) = \frac{3}{2} \sin t - \frac{1}{2} t \cos t$.

(a) $A=\begin{bmatrix}3 & -2\\ 2 & -2\end{bmatrix}$; $\quad |A-\lambda I| = \begin{vmatrix}3-\lambda & -2\\ 2 & -2-\lambda\end{vmatrix} = \ldots = (\lambda+1)(\lambda-2)$

e.w.'n $\lambda_1=-1, \lambda_2=2$

e.v.'n : λ_1 : $\left[\begin{array}{cc|c}3-(-1) & -2 & 0\\ 2 & -2-(-1) & 0\end{array}\right] \sim \left[\begin{array}{cc|c}4 & -2 & 0\\ 2 & -1 & 0\end{array}\right] \sim \left[\begin{array}{cc|c}2 & -1 & 0\\ 0 & 0 & 0\end{array}\right]$

een eigenvector is $\underline{v}_1=\begin{bmatrix}1\\2\end{bmatrix}$

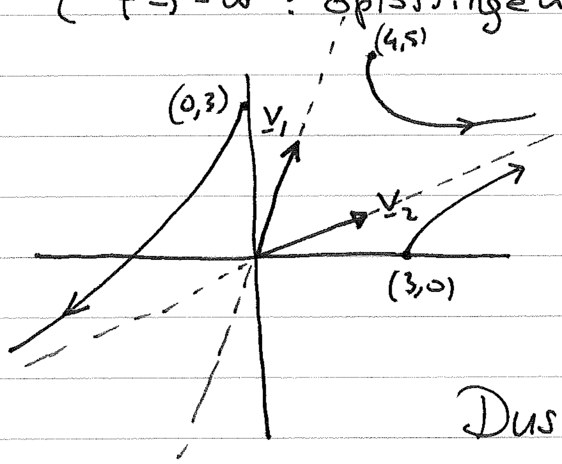
Net zo: e.v. $\underline{v}_2=\begin{bmatrix}2\\1\end{bmatrix}$ bij $\lambda_2=2$

Dan : alg. opl. $\underline{x}(t) = c_1\underline{v}_1\cdot e^{\lambda_1 t} + c_2\underline{v}_2 e^{\lambda_2 t}$

$$= c_1\begin{bmatrix}1\\2\end{bmatrix}e^{-t} + c_2\begin{bmatrix}2\\1\end{bmatrix}e^{2t} = \begin{bmatrix}e^{-t} & 2e^{2t}\\ 2e^{-t} & e^{2t}\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

(b) als $t\to\infty$: oplossingen lopen weg in richting $\pm\begin{bmatrix}2\\1\end{bmatrix}$

($t\to-\infty$: oplossingen 'komen uit' richting $\pm\begin{bmatrix}1\\2\end{bmatrix}$



Bij beginpunt $\begin{bmatrix}4\\5\end{bmatrix}$:

los op : $\begin{bmatrix}e^0 & 2e^0\\ 2e^0 & e^0\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}4\\5\end{bmatrix}$

Dus : $\begin{cases}1c_1+2c_2=4\\ 2c_1+1c_2=5\end{cases} \xrightarrow[\text{rekenen}]{} \begin{matrix}c_1=2\\ c_2=1\end{matrix}$

Antwoord : $\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} = \begin{bmatrix}e^{-t} & 2e^{2t}\\ 2e^{-t} & e^{2t}\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2e^{-t}+2e^{2t}\\ 4e^{-t}+e^{2t}\end{bmatrix}$

(c) Laat $\Phi(t)=\begin{bmatrix}e^{-t} & 2e^{2t}\\ 2e^{-t} & e^{2t}\end{bmatrix}$, dan $\Phi'(t)=\begin{bmatrix}3 & -2\\ 2 & -2\end{bmatrix}\Phi(t)=A\Phi(t)$.

$\underline{x}_p(t)$ via variatie van constante : $\underline{x}_p(t)=\Phi(t)\cdot\underline{c}(t)$

$\underline{x}_p'(t)=A\underline{x}_p(t)+\underline{g}(t) \Rightarrow \cancel{\Phi'(t)\cdot\underline{c}(t)}+\Phi(t)\cdot\underline{c}'(t) = \cancel{A\cdot\Phi(t)\,\underline{c}(t)}+\underline{g}(t)$

$\underline{c}'(t)=\Phi^{-1}(t)\cdot\underline{g}(t)$.

Hier: $\Phi(t) = \begin{bmatrix} e^{-t} & 2e^{2t} \\ 2e^{-t} & e^{2t} \end{bmatrix} \Rightarrow \Phi^{-1}(t) = \dfrac{1}{e^{t} - 4e^{t}} \begin{bmatrix} e^{2t} & -2e^{2t} \\ -2e^{-t} & e^{-t} \end{bmatrix}$

$$= -\frac{1}{3}\begin{bmatrix} e^{t} & -2e^{t} \\ -2e^{-2t} & e^{-2t} \end{bmatrix}$$

dan: $\Phi^{-1}(t)\,g(t) = -\frac{1}{3}\begin{bmatrix} e^{t} & -2e^{t} \\ -2e^{-2t} & e^{-2t} \end{bmatrix} \cdot \begin{bmatrix} e^{-t} \\ 2e^{-t} \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\underline{c}'(t) = \Phi^{-1}(t) \cdot g(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ geeft (bijv.) $\underline{c}(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}$

en dan de gevraagde $\underline{x}_p(t) = \Phi(t) \cdot \underline{c}(t) = \begin{bmatrix} te^{-t} \\ 2te^{-t} \end{bmatrix}$.

3 kritieke punten : $\begin{cases} x' = 0 \\ y' = 0 \end{cases} \Rightarrow \begin{cases} \boxed{y = 0} \\ -x + \mu(1-x^2)y = 0 \rightarrow \boxed{x = 0} \end{cases}$

Lineariseren : Jacobiaan $\begin{bmatrix} 0 & 1 \\ -1-2\mu xy & \mu(1-x^2) \end{bmatrix}$

voor $x = 0, y = 0$: $J = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$

Eigenwaarden : $|J - \lambda I| = \lambda(\lambda - \mu) - 1 \cdot (-1) = \lambda^2 - \mu\lambda + 1$

$|J - \lambda I| = 0$: $\lambda = \dfrac{\mu \pm \sqrt{\mu^2 - 4}}{2}$

$\mu^2 > 4 \Rightarrow$ twee reële eigenwaarden λ_1, λ_2

$\mu > 2 \Rightarrow \mu \pm \sqrt{\mu^2-4} > 0 \Rightarrow \lambda_1 > 0, \lambda_2 > 0$

$\Rightarrow$ instabiele knoop

$\mu < -2 \Rightarrow \lambda_1 < 0, \lambda_2 < 0 \Rightarrow$ stabiele knoop

$\mu^2 < 4 \Rightarrow$ twee complex geconjugeerde e.w.^n $\lambda_{1,2} = \alpha \pm \beta i$

hierbij : $\alpha = \frac{1}{2}\mu$

Dus : $-2 < \mu < 0$: stabiel spiraalpunt

$0 < \mu < 2$: instabiel "

(randgevallen : $\mu = 0$: centrum of spiraalpunt

$\mu = \pm 2$: knoop of spiraalpunt)

4 rustpunten: $\begin{cases} xy(1+y)=0 \Leftrightarrow x=0 \text{ of } y=0 \text{ of } y=-1 \\ xy(1-x)=0 \Leftrightarrow x=0 \text{ of } y=0 \text{ of } x=1. \end{cases}$

Alle punten op de coördinaatassen zijn rustpunten; daarnaast is er nog het punt $(1,-1)$.

b] $\dfrac{y'}{x'} = \dfrac{xy(1-x)}{xy(1+y)} = \dfrac{1-x}{1+y}$ (zolang $x \neq 0$ en $y \neq 0$)

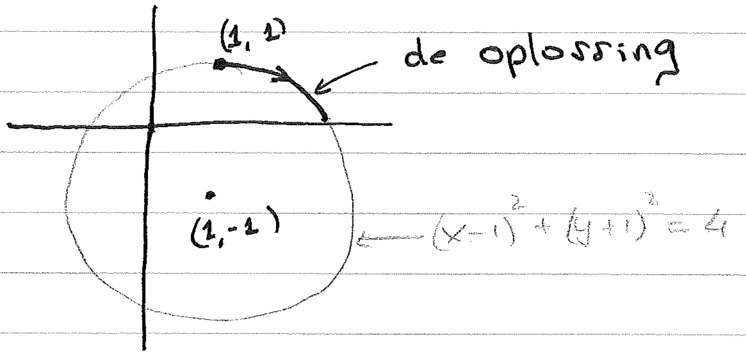
$(1+y)y' = (1-x)x'$: separabele DV

$\int(1+y)\,dy = \int(1-x)\,dx \Longrightarrow y+\frac{1}{2}y^2 = x-\frac{1}{2}x^2+K$

$$\Leftrightarrow \tfrac{1}{2}x^2 - x + \tfrac{1}{2}y^2 + y = K$$
$$\Leftrightarrow x^2-2x+y^2+2y = 2K$$
$$\Leftrightarrow (x-1)^2+(y+1)^2 = 2K+2 = C$$

Dit zijn cirkels met middelpunt $(1,-1)$

c] in het punt $(1,1)$: $x' = 1\cdot1\cdot2=2$, $y'=1\cdot1\cdot0=0$, dus de oplossing loopt vanuit $(1,1)$ naar rechts. De oplossing zal de cirkel met m.p. $(1,-1)$ en straal 2 volgen tot het snijpunt met de x-as, want dit is een rustpunt!



$\boxed{5}$ $c(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{3}\right)$, waarbij

$$a_0 = f_{gem} = \frac{1}{3}\int_0^3 f(x)\,dx = \frac{1}{3}\int_0^2 1\,dx = \frac{2}{3}$$

en voor $k \geq 1$:

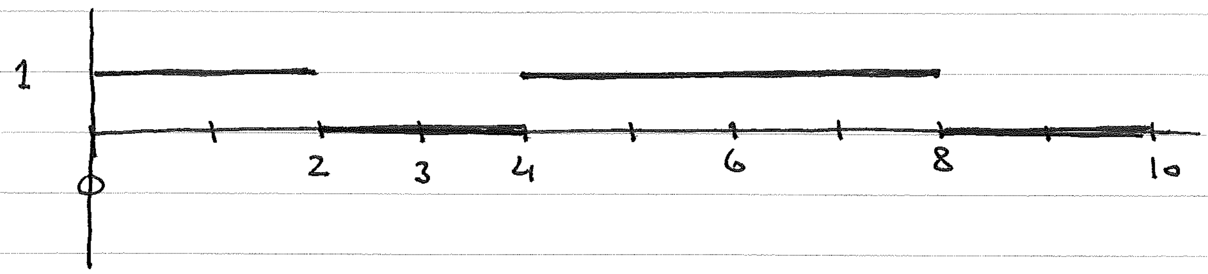
$$a_k = \frac{2}{3}\int_0^2 1 \cdot \cos\frac{k\pi x}{3}\,dx = \frac{2}{3}\cdot\frac{3}{k\pi}\left[\sin\frac{k\pi x}{3}\right]_0^2$$

$$= \frac{2}{k\pi}\cdot\sin\frac{2k\pi}{3} = \frac{2}{k\pi}\cdot\begin{cases} 0 & \text{als } k=3,6,9,\ldots \\ \frac{1}{2}\sqrt{3} & \text{als } k=1,4,7,\ldots \\ -\frac{1}{2}\sqrt{3} & \text{als } k=2,5,8,\ldots \end{cases}$$

Dat geeft:

$$c(x) = \frac{2}{3} + \frac{\sqrt{3}}{\pi}\cdot\left[\sin x - \frac{\sin(2x)}{2} + \frac{\sin(4x)}{4} - \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7} - \frac{\sin(8x)}{8} + \ldots\ldots\right]$$

(b) $c(x)$ is de <u>even</u>, <u>periodieke</u> voortzetting van $f(x)$



$\boxed{6}$ stel $u(x,t) = X(x)T(t)$

$$u_{tt} = 4u_{xx} \longrightarrow X(x)\cdot T''(t) = 4X''(x)\cdot T(t)$$

$$\longrightarrow \frac{T''(t)}{T(t)} = 4\frac{X''(x)}{X(x)}, \text{ kan alleen als } \frac{X''(x)}{X(x)} = c$$

verder: $u(0,t) = 0 \longrightarrow X(0) = 0$

$u(2,t) = 0 \longrightarrow X(2) = 0$

$$\frac{X''(x)}{X(x)} = c \Rightarrow X''(x) - cX(x) = 0$$

2^e orde lineaire DV!

3 gevallen: (I) $c = \lambda^2 > 0 \Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x}$

$X(0) = X(2) = 0 \longrightarrow A = B = 0$. geen opl^n

(II) $c = 0 \Rightarrow X(x) = A + Bx$

randvoorw^n geven opnieuw $A = B = 0$.

(III) $c = -\lambda^2$: $X''(x) + \lambda^2 X(x) = 0 \Rightarrow X(x) = A\sin(\lambda x) + B\cos(\lambda x)$

$x = 0$: $0 + B = 0 \longrightarrow B = 0$

$x = 2$: $A\cdot\sin(2\lambda) = 0 \longrightarrow 2\lambda = k\pi,\ k \in \mathbb{Z}$

$\longrightarrow \lambda = \frac{k\pi}{2},\ k = 1, 2, 3, \ldots$

(vanwege $\sin(-x) = -\sin(x)$ geven negatieve k dezelfde opl^n als positieve k)

voor deze λ: $c = -\left(\frac{k\pi}{2}\right)^2$

$$\frac{T''(t)}{T(t)} = -\left(\frac{k\pi}{2}\right)^2\cdot 4 \Rightarrow T(t) = C_k\cdot\sin(k\pi t) + D_k\cdot\cos(k\pi t)$$

$u_t(x,0) = 0 \Rightarrow T'(0) = 0 \Rightarrow C_k = 0$

$\Rightarrow T(t) = D_k\cos(k\pi t)$

Al met al: $u(x,t) = \sum_{k=1}^{\infty} D_k \sin\left(\frac{k\pi}{2}x\right)\cdot\cos(k\pi\cdot t)$

$t = 0$ invullen: $\sum_{k=1}^{\infty} D_k \sin\left(\frac{k\pi}{2}x\right) = x\cdot(2-x) = 2x - x^2 = g(x)$

coëff^n van sinusreeks van $g(x)$!

$$D_k = \frac{2}{2}\int_0^2 (2x - x^2)\,\sin\left(\tfrac{k\pi}{2}x\right) dx$$

$$= 2\int_0^2 x \sin\left(\tfrac{k\pi}{2}x\right) dx - \int_0^2 x^2 \sin\left(\tfrac{k\pi}{2}x\right) dx$$

$$\underset{(\text{tabel})}{=} \left[2\cdot\frac{2^2}{k^2\pi^2}\sin\left(\tfrac{k\pi}{2}x\right) - 2\cdot\frac{2x}{k\pi}\cos\left(\tfrac{k\pi}{2}x\right) - \left(2\cdot\frac{2^3}{k^3\pi^3} - \frac{2x^2}{k\pi}\right)\cos\left(\tfrac{k\pi}{2}x\right) - 2x\cdot\frac{2^2}{k^2\pi^2}\sin\left(\tfrac{k\pi}{2}x\right)\right]_{x=0}^{2}$$

$$= -\frac{8}{k\pi}\cos(k\pi) - \left(\frac{16}{k^3\pi^3} - \frac{8}{k\pi}\right)\cos(k\pi) + \frac{16}{k^3\pi^3} = \dots = \begin{cases} \frac{32}{k^3\pi^3} & \text{als } k \text{ oneven} \\ 0 & \text{als } k \text{ even.} \end{cases}$$

<u>Dus</u> $$2x - x^2 \sim \sum_{k \text{ oneven}} \frac{32}{k^3\pi^3}\sin\left(\tfrac{k\pi}{2}x\right)$$

<u>en</u>

$$u(x,t) = \sum_{k \text{ oneven}} \frac{32}{\pi^3}\cdot\frac{1}{k^3}\sin\left(\tfrac{k\pi}{2}x\right)\cdot\cos(k\pi t)$$

$$= \frac{32}{\pi^3}\sum_{k \text{ oneven}} \frac{\sin\left(\tfrac{k\pi}{2}x\right)\cdot\cos(k\pi\cdot t)}{k^3}$$