Tentamen Differentiaalvergelijkingen 2 nov. 2001

Opgave 1

$$\begin{cases} \dfrac{dy}{dt} + 2y = e^{-t} \quad , \ t > 0, \\ y(0) = 1 \end{cases}$$

Bepaal m.b.v variatie der constanten de opl. van dit beginwaardeprobleem voor $y = y(t)$.

1. $\dfrac{dy}{dt} + 2y = 0$

$$\frac{dy}{dt} = -2y$$

$$\frac{dy}{dt} \cdot \frac{1}{y} = -2$$

$$\ln|y| = -2t + k$$

$$|y| = e^{-2t} \cdot e^{k}$$

$$y = c \cdot e^{-2t}$$

2. $y(t) = c(t) \cdot e^{-2t}$

$$\frac{dc(t)}{dt} e^{-2t} - 2ce^{-2t} + 2ce^{-2t} = e^{-t}$$

$$\frac{dc(t)}{dt} e^{-2t} = e^{-t}$$

$$\frac{dc(t)}{dt} = e^{t}$$

$$c(t) = e^{t} + A$$

$$y(t) = (e^{t} + A)e^{-2t} = e^{-t} + Ae^{-2t}$$

Opgave 2

$$\begin{cases} \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = \delta(t-3) \quad , \quad t>0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

Bepaal m.b.v. transformatie van Laplace de oplossing van dit beginwaarde probleem voor $y = y(t)$

$$\begin{cases} y'' + 2y' + 3y = \delta(t-3) \\ y'(0) = 1 \quad ; \quad y(0) = 0 \end{cases} \qquad \mathcal{L}\{y(t)\} = Y(s)$$

17 + 18. $s^2 Y(s) - s\,y(0) - y'(0) + 2\{s\,Y(s) - y(0)\} + 3\,Y(s) = e^{-3s}$

$$\{s^2 + 2s + 3\}\,Y(s) - 1 = e^{-3s}$$

$$Y(s) = \frac{e^{-3s} + 1}{s^2 + 2s + 3} = \frac{e^{-3s}}{(s+1)^2 + 2} + \frac{1}{(s+1)^2 + 2}$$

9. $\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{(s+1)^2 + 2} \xrightarrow{\mathcal{L}^{-1}} \frac{1}{\sqrt{2}}\, e^{-t} \sin \sqrt{2}\, t$

13. $y = \mathcal{L}^{-1}(Y(s)) = \frac{1}{\sqrt{2}}\, u_3\, e^{-(t-3)} \sin(\sqrt{2}(t-3)) + \frac{1}{\sqrt{2}}\, e^{-t} \sin(\sqrt{2}\, t)$

<u>Opgave 3</u>

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + x_2 \\ \dfrac{dx_2}{dt} = 4x_1 + x_2 \end{cases}$$

a. Bepaal de alg. opl. van dit stelsel DV'en.

$$x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$$

• Bepaal determinant $(A - r I) = 0$

$$\left| \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix} \right| = 0$$

$(1-r)(1-r) - (4 \cdot 1) = r^2 - 2r - 3$

abc - form. : $r = 3 \lor -1$

• Eigenvectoren $r = 3$:

$(A - 3I)\xi = 0$

$$\left| \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \right| \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \qquad \begin{cases} -2\xi_1 + \xi_2 = 0 \\ 4\xi_1 + \xi_2 = 0 \end{cases} \Rightarrow \xi_2 = 2\xi_1$$

$\xi_1 = c_1 \qquad \xi_2 = 2c_1$

$$\Rightarrow \text{Een opl. is : } \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{3t} = \begin{pmatrix} c_1 \\ 2c_1 \end{pmatrix} e^{3t} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

• Eigenvectoren $r = -1$:

$(A + I)\xi = 0$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \qquad \begin{cases} 2\xi_1 + \xi_2 = 0 \\ 4\xi_1 + 2\xi_2 = 0 \end{cases} \Rightarrow 2\xi_1 = -\xi_2$$

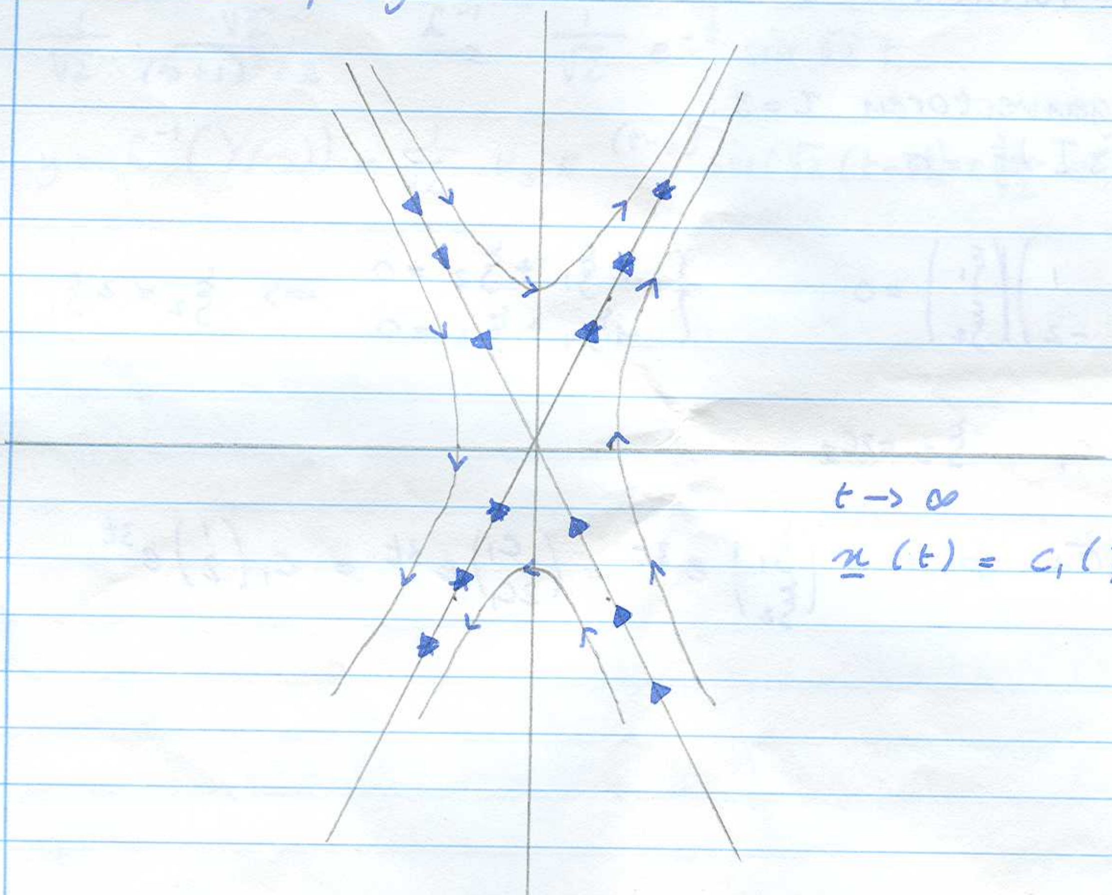
$\xi_1 = c_2 \qquad \xi_2 = -2\xi_1 = -2c_2$

$\Rightarrow$ Een opl. is: $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{-t} = \begin{pmatrix} c_2 \\ -2c_2 \end{pmatrix} e^{-t} = c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$

$\Rightarrow$ Een algemene oplossing is:

$$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

b. Schets in het fase-vlak ((x_1, x_2)-vlak) enkele intergraalkrommen. Wat voor type evenwichtspunt is de oorsprong?



$t \to \infty$

$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$

Opgave 4

$$\begin{cases} \dfrac{dx}{dt} = 1 - y, \\ \dfrac{dy}{dt} = x^2 - y^2. \end{cases}$$

a. Bepaal de evenwichtspunten v/h stelsel DV'en.

$1 - y = 0 \rightarrow y = 1$
$x^2 - y^2 = 0 \rightarrow x = 1 \vee -1$

Evenwichtspunten: $(1,1)$ en $(-1,1)$

b. Bepaal de ~~eigenwaarden~~ gelineariseerde stelsels DV'en in de omgeving van elk kritiek punt.

$(-1,1)$: $x = -1 + \tilde{x}$ met $\tilde{x}$ en $\tilde{y}$ klein
$y = 1 + \tilde{y}$

Invullen in DV:

$$\begin{cases} \dfrac{d(-1+\tilde{x})}{dt} = 1 - (1+\tilde{y}) \\ \dfrac{d(1+\tilde{y})}{dt} = (-1+\tilde{x})^2 - (1+\tilde{y})^2 \end{cases} \Longrightarrow$$

$$\frac{d\tilde{x}}{dt} = -\tilde{y}$$

$$\frac{d\tilde{y}}{dt} = -2\tilde{x} - 2\tilde{y} + \tilde{x}^2 - \tilde{y}^2$$

$\Longrightarrow$ $\tilde{x}^2$ en $\tilde{y}^2$ zijn zeer klein t.o.v. $\tilde{x}$ en $\tilde{y}$: verwaarl.baar.

Lineaire stelsel:

$$\tilde{x} = \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

unstable saddle point.

$p = a_{11} + a_{22} = 0 - 2 = -2$
$q = \det(A) = a_{11}a_{22} - a_{12}a_{21} = (0 \cdot -2) - (-2 \cdot -1) = -2$

$(1,1)$: $x = 1 + \tilde{x}$

$y = 1 + \tilde{x}$

Invullen in DV:

$$\begin{cases} \dfrac{d(1+\tilde{x})}{dt} = 1 - (1+\tilde{y}) \\ \dfrac{d(1+\tilde{y})}{dt} = (1+\tilde{x})^2 - (1+\tilde{y})^2 \end{cases}$$

$$\frac{d\tilde{x}}{dt} = -\tilde{y}$$

$$\frac{d\tilde{y}}{dt} = 2\tilde{x} - 2\tilde{y} + \tilde{x}^2 - \tilde{y}^2$$

Lineaire Stelsel:

$$\tilde{x} = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

$p = 0 - 2 = -2$

$q = (0 \cdot -2) - (-1 \cdot 2) = 2$

$\Longrightarrow$ asymptosich stabiel spiraal punt.

c. Bepaal de eigenwaarden van de in b. gevonden stelsels DV'en. Welke conclusies kunt u trekken over het niet-lineaire stelsel DV'en?

$(-1,1)$: Lineaire stelsel : $\tilde{x}' = \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$

$p = -2$

$q = -2$

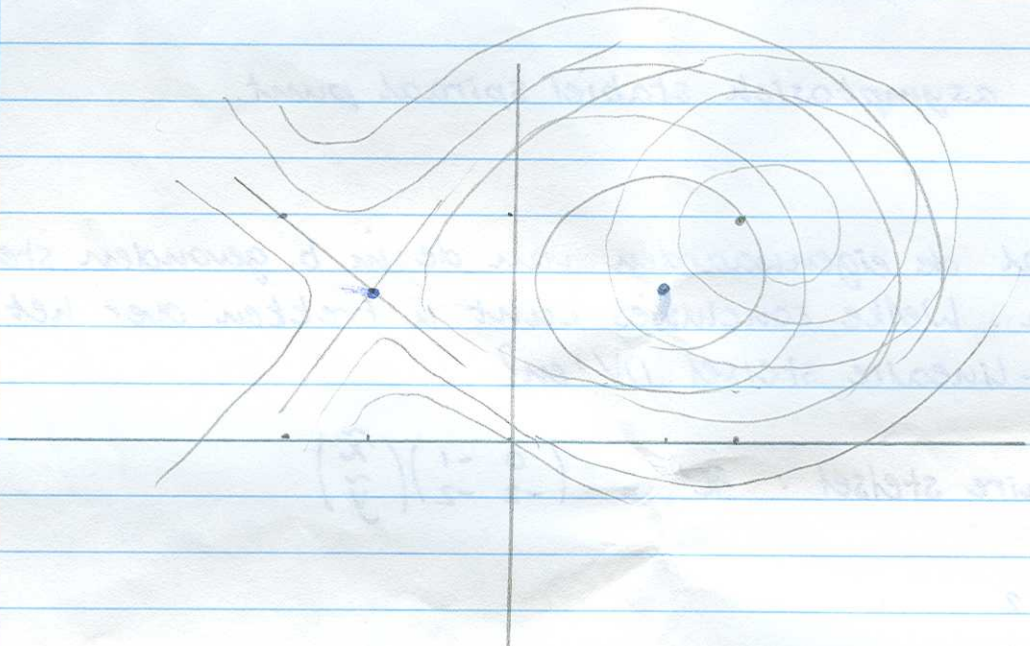
$$\left. \begin{array}{l} x' = A(x(t)) \\ \text{Eigenvalue} = (A - \lambda I)x = 0 \end{array} \right\} \Rightarrow \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0-\lambda & -1 \\ -2 & -2-\lambda \end{pmatrix} \rightarrow (0-\lambda)(-2-\lambda) - (-2 \cdot -1) = \lambda^2 + 2\lambda - 2 = 0$$

λ uitrekenen mbv abc

(1,1) : zie (-1,1).

d. Schets in het fasevlak enkele baankrommen van het niet-lineaire stelsel DV'en.



Opgave 5.

a. Bepaal de Fourierreeks voor f.

Fourierreeks: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \{ a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right) \}$

$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0$, want f is even

$2L = 4\pi \rightarrow L = 2\pi$

$a_0 = \frac{1}{L}\int_0^{\pi} f(x)\, dx = \frac{1}{2\pi}\int_0^{\pi} x\, dx = \frac{1}{2\pi}\left(\frac{1}{2}\pi^2\right) = \frac{\pi}{4}$

$a_n = \frac{2}{L}\int_0^{2\pi} x \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{\pi}\int_0^{2\pi} x\cos\left(\frac{n\pi x}{2\pi}\right) dx$

$= \frac{1}{\pi}\left[\frac{2}{n} x \sin\frac{nx}{2} + \frac{4}{n^2}\cos\left(\frac{nx}{2}\right)\right]_0^{2\pi}$

$= \frac{1}{\pi}\left(0 + \frac{4}{n^2}\cos n\pi - \{0 + \frac{4}{n^2}\}\right)$

$= \frac{1}{\pi}\left(\frac{4}{n^2}\cos n\pi - \frac{4}{n^2}\right) \Longrightarrow$ $= 0$ voor $n = 0, 2, 4 \ldots$
$= -\frac{8}{\pi n^2}$ voor $n = 1, 3, 5 \ldots$

Dus Fourierreeks: $f(x) = \frac{\pi}{8} - 8\sum_{n=1,3,5\ldots}^{\infty} \frac{1}{\pi n^2}\cos\left(\frac{nx}{2}\right)$

b.

$\int x \sin nx = \left(\frac{1}{n}\right)^2 [\sin(nx) - nx\cos(nx)]$

$\int x \cos nx = \frac{1}{n} x \sin(nx) + \left(\frac{1}{n}\right)^2 \cos(nx)$

Opgave 6.

$$\begin{cases} u_{xx} = u_t & 0 < x < \pi, \quad t > 0 \\ u(x,0) = \sin x + 5\sin 3x, & 0 < x < \pi \\ u(0,t) = u(\pi,t) = 0, & t \geq 0 \end{cases}$$

Stap 1: Invullen in de PDV: $XT' = X''T$

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda \rightarrow \begin{cases} X'' + \lambda X = 0 \\ T' = \lambda T \end{cases}$$

Stap 2: Invullen in de RVW^n:

$u(0,t) = 0 \Rightarrow X(0)\,T(t) = 0$

$X(0) = 0 \qquad T(t) \neq 0$

$u(\pi,t) = 0 \Rightarrow X(\pi)\,T(t) = 0$

$X(\pi) = 0 \qquad T(t) \neq 0$

Stap 3: Bepaal $X(x)$ met:

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(\pi) = 0 \end{cases}$$

Onderscheid de 3 volgende gevallen:

- 1. $\lambda < 0$:

$X'' + \lambda X = 0 \xrightarrow{e^{rx}}$ kar. vgl. $r^2 + \lambda = 0$

$\rightarrow r^2 = -\lambda > 0 \rightarrow r = \sqrt{-\lambda}$ of $-\sqrt{-\lambda}$

$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$

$= k_1 \sinh(\sqrt{-\lambda}x) + k_2 \cosh(\sqrt{-\lambda}x)$

$X(0) = 0 \rightarrow k_1 \overset{0}{\sinh}(0) + k_2 \overset{1}{\cosh}(0) = 0 \rightarrow k_2 = 0$

$X(\pi) = 0 \rightarrow k_1 \sinh(\sqrt{-\lambda}\pi) + 0 = 0 \rightarrow k_1 = 0$

$\longrightarrow$ alg. opl. : $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin(nx)$ met c_n nader te bepalen.

Stap 6 : Pas de alg. opl. aan aan de beginvoorwaarden.

$u(x,0) = f(x) = \sin x + 5 \sin 3x$

$u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx) = \sin x + 5 \sin 3x = \sum_{n=1}^{\infty} f_n \sin(nx)$

$c_n = f_n$

$f_1 = 1 \quad , \quad f_2 = 0 \quad , \quad f_3 = 5 \quad , \quad f_n = 0 \quad \text{voor } n \geq 4$

$\longrightarrow u(x,t) = e^{-t} \sin x + 5 e^{-9t} \sin 3x.$