Technische Universiteit Delft, Fac. EWI

Exam Differential Equations, AESB2110, 4 November 2015, 13:30-16.30

Each answer must be clearly motivated.
You receive a table with Laplace transformations and a few integrals. You may use a simple calculator (which actually you won't need.).
The (maximum) scores: exc.1: **13** pt; exc.2: **10** pt; exc.3: **8** pt; exc.4: **9** pt;

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1. **a.** Show that the Laplace transform of $f(t) = t\cos t$ equals $\dfrac{1}{s^2+1} - \dfrac{2}{(s^2+1)^2}$.

b. Use the above to find the inverse Laplace transform of $G(s) = \dfrac{1}{(s^2+1)^2}$.

c. i. Using the Laplace transform find the solution of the initial value problem

$$\begin{cases} y''(t) + 4y(t) = \delta(t - \frac{1}{4}\pi) - \delta(t - \pi) \\ y(0) = 2, \;\; y'(0) = 0 \end{cases}.$$

ii. Write $y(t)$, for $t \geq \pi$, in its most simplified form.

d. Check which of the functions $y_1(t) = t^3$, $y_2(t) = t^2$ and $y_3(t) = 1/t$ are homogeneous solutions of the differential equation $t^2y''(t) - ty'(t) - 3y(t) = 2t^3$.

Find a solution of this (inhomogeneous) differential equation using variation of parameters.

2. **a.** Find the value(s) of β for which $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & \beta \\ 1 & -1 & 1 & -1 \end{vmatrix} = 8.$

Work accurately here. (If you make more than 1 calculation error, the credits will be 0 points.)

For the next three questions the matrix A is given by $\begin{bmatrix} 1 & 2 & 4 \\ 2 & -2 & 2 \\ 4 & 2 & 1 \end{bmatrix}$.

b. Which of the two vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$ is/are eigenvectors of A?

c. For this exercise you **don't need** to find the characteristic equation of the matrix first! Which of the values $\lambda_1 = 5$ and $\lambda_2 = -3$ is/are eigenvalues of A?
In the case of an eigenvalue: give a basis for the corresponding eigenspace.

d. Check whether A is diagonalizable. (Give an argument!)
If A **is** diagonalizable: find P and D such that $A = PDP^{-1}$.

3. In an isolated nature reserve there are two species with (scaled) population sizes $x(t)$ and $y(t)$. The growth model is given by the following two differential equations involving $x(t)$ and $y(t)$

$$\frac{dx}{dt} = 0.5x\,(6 - x - 2y), \qquad \frac{dy}{dt} = 0.25y\,(-2 + x - 2y)$$

where t is measured in years. Because of the context for the exercise we only consider the first quarter: $x \geq 0,\ y \geq 0$.

a. Is this a predator-prey model or a model with competing species? (Of course you have to motivate your answer.)

b. Find all stationary points. (Do this carefully, since also the next part depends on the answer. As a check: there are three points where both coordinates are nonnegative.)

c. Find the local behavior around the stationary points.
Classify them as node, star point, etc, and decide whether they are stable or unstable.

d. Sketch the local behaviour in the phase plane. For the nodes and saddle points (if there are any): make clear what is the role of the eigenvectors.
In the same picture sketch the trajectories that start from the points $(0,5)$ and $(1,5)$.

4. **a.** Using the method of separation of variables (so no ready made solutions!) find the solution of the following partial differential equation with boundary values:

$$\begin{cases} 4\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & \text{(I)} \\ u(0,y) = u(4,y) = 0 & \text{(II)} \\ u(x,0) = 0 & \text{(III-a)} \\ u(x,2) = h(x) & \text{(III-b)} \end{cases}$$

in the domain $D\colon\ 0 \leq x \leq 4,\ \ 0 \leq y \leq 2$.

b. What will be the solution if $h(x) = \sin(\pi x)$?

c. What will be the solution if we replace (III-a) by $u(x,0) = -\sin(\pi x)$, while keeping the other boundary values the same ?

(If you couldn't solve **b.** you may describe how you can adapt the solution of **a.** if condition (III) is changed to $u(x,2) = h(x),\ u(x,0) = -h(x)$.)

Solutions

1a Use the table (13 is your lucky number!): $\mathcal{L}[\cos t] = \dfrac{s}{s^2+1} \quad \Rightarrow$

$$\Rightarrow \quad \mathcal{L}[t\cos t] = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right] = -\left[\frac{s^2+1-s\cdot 2s}{(s^2+1)^2}\right] = -\left[\frac{-(s^2+1)+2}{(s^2+1)^2}\right] =$$

$$= \quad -\left[\frac{-(s^2+1)}{(s^2+1)^2} + \frac{2}{(s^2+1)^2}\right] = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}$$

1b From **a.** it follows that $\mathcal{L}[-\frac{1}{2}t\cos t] = -\dfrac{1}{2}\dfrac{1}{s^2+1} + \dfrac{1}{(s^2+1)^2}$.

To get rid of the term $-\dfrac{1}{2}\dfrac{1}{s^2+1}$, one can add a term $\frac{1}{2}\sin t$ in the t-domain,

giving the answer $\quad \mathcal{L}^{-1}\left[\dfrac{1}{(s^2+1)^2}\right] = -\frac{1}{2}t\cos t + \frac{1}{2}\sin t$

1(c)i $s^2Y(s) - 2s + 4Y(s) = e^{-\frac{1}{4}\pi s} - e^{-\pi s} \Leftrightarrow Y(s) = \dfrac{2s}{s^2+4} + \dfrac{e^{-\frac{1}{4}\pi s}}{s^2+4} - \dfrac{e^{-\pi s}}{s^2+4}$
is quickly transformed back term by term, leading to the
answer: $\quad y(t) = 2\cos 2t + \frac{1}{2}u_{\frac{1}{4}\pi}(t)\sin(2(t-\frac{1}{4}\pi)) - \frac{1}{2}u_\pi(t)\sin(2(t-\pi))$

1(c)ii For $t \geq \pi$, both step functions give 1, so

$$\begin{aligned} y(t) &= 2\cos 2t + \tfrac{1}{2}\sin(2(t-\tfrac{1}{4}\pi)) - \tfrac{1}{2}\sin(2(t-\pi)) \\ &= 2\cos 2t + \tfrac{1}{2}\sin(2t-\tfrac{1}{2}\pi) - \tfrac{1}{2}\sin(2t-2\pi) \\ &= 2\cos 2t + \tfrac{1}{2}\cos(2t) - \tfrac{1}{2}\sin(2t) = \tfrac{5}{2}\cos(2t) - \tfrac{1}{2}\sin(2t) \end{aligned}$$

1d $t^2y_1''(t) - ty_1'(t) - 3y_1(t) = 6t^3 - 3t^3 - 3t^3 = 0,$
$t^2y_2''(t) - ty_2'(t) - 3y_2(t) = 2t^2 - 2t^2 - 3t^3 \neq 0,$ and
$t^2y_3''(t) - ty_3'(t) - 3y_3(t) = 2t^{-1} + t^{-1} - 3t^{-1} = 0,$
so y_1 and y_3 are homogeneous solutions and y_2 isn't.

To find a particular solution: put $y_P(t) = u_1(t)y_1(t) + u_2(t)y_3(t) = u_1(t)t^3 + u_2(t)t^{-1}$, plugging this into the DE, and forcing $u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$ halfway, to avoid second order derivatives of u_1, u_2 yields

$$\begin{cases} u_1'(t)t^3 + u_2'(t)t^{-1} & = 0 \\ t^2\big(3t^2\,u_1'(t) - t^{-2}u_2'(t)\big) & = 2t^3 \end{cases} \Leftrightarrow \begin{cases} t^3\,u_1'(t) + t^{-1}u_2'(t) & = 0 \\ 3t^4\,u_1'(t) - u_2'(t) & = 2t^3 \end{cases}$$

Subtracting the first equation $3t$ times from the second gives

$$-4u_2'(t) = 2t^3 \;\Rightarrow u_2'(t) = -\tfrac{1}{2}t^3 \;\Rightarrow u_2(t) = -\tfrac{1}{8}t^4\,(+C).$$

Adding t times the first equation to the second gives

$$4t^4\,u_1'(t) = 2t^3 \;\Rightarrow u_1'(t) = 1/2t^{-1} \;\Rightarrow u_1(t) = \tfrac{1}{2}\ln t\,(+C_2).$$

As a particular solution we find

$$y_P(t) = \tfrac{1}{2}\ln t\cdot t^3 - \tfrac{1}{8}t^4\cdot t^{-1} = \tfrac{1}{2}t^3\ln t - \tfrac{1}{8}t^3$$

The second term is a homogeneous solution, thus may be dropped.

2a

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & \beta \\ 1 & -1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & \beta-1 \\ 0 & -2 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & \beta-7 \\ 0 & 0 & 4 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & \beta-7 \\ 0 & 4 & 4 \end{vmatrix} = \begin{vmatrix} 0 & \beta-7 \\ 4 & 4 \end{vmatrix} = -4(\beta-7)$$

So the determinant equals 8 if $-4(\beta-7) = 8 \Leftrightarrow \underline{\underline{\beta = 5}}$.

2b $A\mathbf{v}_1 = \begin{bmatrix} 12 \\ 6 \\ 12 \end{bmatrix} = 6\mathbf{v}_1$, so $\mathbf{v}$ is an eigenvector for eigenvalue 6.
Likewise $A\mathbf{v}_2 = (-3)\mathbf{v}_2$, so $\mathbf{v}_2$ is an eigenvctor for eigenvalue -3.

2c Note: λ is an eigenvalue iff $\text{Det}(A-\lambda I) = 0$, and $A\mathbf{v} = \lambda\mathbf{v}$ is equivalent to $(A-\lambda I)\mathbf{v} = \mathbf{0}$.

$[\,A - 5I\,] \sim \begin{bmatrix} \boxed{-4} & 2 & 4 \\ 2 & -7 & 2 \\ 4 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} -4 & 2 & 4 \\ 0 & -6 & 4 \\ 0 & 4 & 0 \end{bmatrix}$ is obviously a matrix of rank 3. So the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ only has the trivial solution, and 5 is **not** an eigenvalue. Likewise $[\,A - (-3)I\,|\,\mathbf{0}] \sim \begin{bmatrix} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has two independent solutions, e.g. $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$, and $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a (possible) basis of eigenvectors.

2d Diagonalizable is equivalent to: there is a basis of eigenvectors. For a 3×3 matrix this means: there are three independent eigenvectors. These were already found above: $\{\mathbf{v}_1, \mathbf{u}_1, \mathbf{u}_2\}$.
Remark: $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}_1\}$ or $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}_2\}$ are also possible, but $\{\mathbf{v}_2, \mathbf{u}_1, \mathbf{u}_2\}$ is dependent.
We then have (for instance)

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 1 \\ -1 & 1 & 2 \end{bmatrix}^{-1}$$

3a The interaction follows from the coefficients of the terms xy. In the equation for $x'(t)$ this coefficient equals -1, so the influence of y on x is negative, the other coefficient is positive, from which we can conclude that y are the predators and x the preys.

3b We have to find the solutions of:

$$\begin{cases} 0.5x\,(6-x-2y) &= 0 \\ 0.25y\,(-2+x-2y) &= 0 \end{cases} \longleftrightarrow \begin{cases} x = 0 \text{ or } 6-x-2y = 0 \\ y = 0 \text{ or } -2+x-2y = 0 \end{cases}$$

Combining the four (!) possibilities easily produces the three points $(0,0)$, $(0,-1)$ and $(6,0)$, and from $\begin{cases} 6-x-2y = 0 \\ -2+x-2y = 0 \end{cases}$ follows the fourth point $(4\,1)$.
The point $(0,-1)$ is not relevant: population sizes cannot be negative.

3c $\begin{cases} \dfrac{dx}{dt} = 0.5x\,(6-x-2y) \\ \dfrac{dy}{dt} = 0.25y\,(-2+x-2y) \end{cases} \Longrightarrow \begin{cases} \dfrac{dx}{dt} = 3x - 0.5x^2 - xy \\ \dfrac{dy}{dt} = -0.5y + 0.25xy - 0.5y^2 \end{cases}$

For the linearizations (needed for the local behavior) we need the Jacobian matrix:

$$J(x,y) = \begin{bmatrix} 3-x-y & -x \\ 0.25y & -0.5+0.25x-y \end{bmatrix}.$$

$J(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & -0.5 \end{bmatrix}$ has eigenvalues 3 and -0.5, so (0,0) is a **saddle point** (**unstable**).

Eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for $\lambda = 3$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $\lambda = -0.5$.

$J(6,0) = \begin{bmatrix} -3 & -6 \\ 0 & 1 \end{bmatrix}$ has eigenvalues -3, 1, so (6,0) is also a **saddle point** (**unstable**).

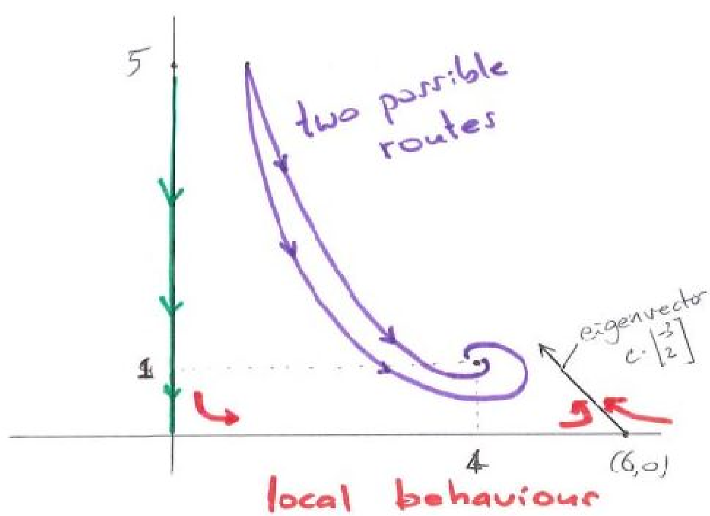
Eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for $\lambda = -3$ and $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ for $\lambda = 1$.

$J(4,1) = \begin{bmatrix} -2 & -4 \\ 0.25 & -0.5 \end{bmatrix}$ gives some more work:

the characteristic polynomial: $\begin{vmatrix} -2-\lambda & -4 \\ 0.25 & -0.5-\lambda \end{vmatrix} = \lambda^2 + 2.5\lambda + 2$ gives the eigenvalues $\dfrac{-2.5 \pm i\sqrt{1.75}}{2}$, with negative real parts, so $(4,1)$ is a **stable spiral point**.

For a sketch: a solution that start from a point on the (positive) y-axis will remain on this axis (since $x' = 0$) and, according to the direction field will approach the origin. The solution from (1,5) will go down at first, turn left (keeping (0,0) at its right) and eventually converge to the only stable stationary point.

3d



4a The usual three-step approach:
First put $u(x,t) = X(x)Y(y)$,
next: separate the variables:

$$4\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \sigma \ \text{(constant)} \to \begin{cases} X'' + \sigma X &= 0 \\ Y'' - 4\sigma Y = 0 \end{cases}$$

The boundary values $u(0,y) = X(0)Y(y) = 0$, $u(4,y) = X(4)Y(y) = 0$
imply $X(0) = X(4) = 0$.
Three cases to consider: $\sigma = \rho^2 > 0$, $\sigma = 0$, $\sigma = -\rho^2 < 0$, ... (on the exam the student is required to work this out completely)

$$\left\{ \begin{array}{l} X'' + \sigma X = 0 \\ X(0) = 0, \ \ X(4) = 0 \end{array} \right\} \Rightarrow \ldots \Rightarrow \text{for } \sigma = -\left(\frac{n\pi}{4}\right)^2 : \quad \begin{array}{l} X_n(x) = B_n \sin\left(\frac{n\pi}{4}x\right), \\ \text{for } n = 1,\, 2,\, \ldots. \end{array}$$

For the values of σ just found:

$$Y'' - 4(\frac{n\pi}{4})^2 Y = 0, \ Y(0) = 0 \ \Rightarrow \ \ldots \ \Rightarrow \quad Y_n(y) = A_n\left[e^{\frac{n\pi}{2}y} - e^{-\frac{n\pi}{2}y}\right] \ \text{ or: } Y_n(y) = A_n \sinh\left(\frac{n\pi}{2}y\right)$$

All in all

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{4}x\right) \sinh\left(\frac{n\pi}{2}y\right)$$

4b $h(x) = \sin(\pi x)$ means that we want to achieve

$$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{4}x\right) \sinh\left(\frac{n\pi}{2}2\right) = \sin(\pi x)$$

Or, using the ... notation

$$C_1 \sin\left(\tfrac{1}{4}\pi x\right)\sinh(\pi) + C_2 \sin\left(\tfrac{2}{4}\pi x\right)\sinh(2\pi) + C_3 \sin\left(\tfrac{3}{4}\pi x\right)\sinh(3\pi) + C_4 \sin(\pi x)\sinh(4\pi) + \ldots = \\ = \sin(\pi x)$$

from which we may conclude that $C_4 \sinh(4\pi) = 1$ and all another C_n are 0.
This gives the **final answer**: $u(x,y) = \dfrac{1}{\sinh(4\pi)} \sin(\pi x) \sinh\left(\frac{4\pi}{2}y\right) = \dfrac{\sin(\pi x)\sinh(2\pi y)}{\sinh(4\pi)}$.

4c First, to find a solution that satisfies $u(x,0) = -\sin(\pi x)$ and $u(x,2) = \mathbf{0}$, one simply replaces y by $2-y$ in the solution found above, and puts a minus in front.

To have a solution of the PDE that satisfies both inhomogeneous boundary conditions: add the two:

$$u(x,y) = \frac{\sin(\pi x)\sinh(2\pi y)}{\sinh(4\pi)} - \frac{\sin(\pi x)\sinh(2\pi(2-y))}{\sinh(4\pi)} = \sin(\pi x)\frac{\sinh(2\pi y) - \sinh(2\pi(2-y))}{\sinh(4\pi)}$$