Technische Universiteit Delft, Fac. EWI

Exam Differential Equations, AESB2110, 7 November 2014, 14:00-17.00

Each answer must be clearly motivated.
You receive a table with Laplace transformations and a few integrals. You may use a simple calculator (which actually you won't need.).
The (maximum) scores: exc.1: **6** pt; exc.2: **6** pt; exc.3: **9** pt; exc.4: **9** pt; exc.5: **6** pt.

1. **a.** Find the Laplace transform of the function $\begin{cases} t, & \text{als } 0 \le t \le 2 \\ 2\ , & \text{als } t \ge 2 \end{cases}$

b. *Using the Laplace transform* find the solution of the initial value problem $y''(t) + 4y'(t) + 4y(t) = t^2 e^{-2t},\ \ y(0) = 0,\ \ y'(0) = 2.$

2. We consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & a \end{bmatrix}$, where a is a real number.

a. Show that A has the eigenvalue 0 for *every* value of a.

b. Find all eigenvalues of A for the case where $a = -2$.

c. Give the definition of 'diagonalizable matrix' (this does **not** contain the word eigenvector!), and give an equivalent characterization in terms of eigenvalues and/or eigenvectors.

d. For which value(s) of a is the matrix $\begin{bmatrix} 1 & 2 & a & 4 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ diagonalizable?

3. In an isolated nature reserve there are two species with (scaled) population sizes $x(t)$ and $y(t)$. The growth model is given by the two differential equations $x(t)$ en $y(t)$

$$\frac{dx}{dt} = 0.5x\,(6 - x - 2y), \qquad \frac{dy}{dt} = 0.25y\,(8 - 2x - 2y)$$

where t is measured in years.

a. Is this a predator-prey model or a model with competing species? (Of course you have to motivate your answer.)

b. Using the direction field show that solutions starting from a point inside the square $0 \le x \le 10,\ 0 \le y \le 10$ will never leave this square.

c. Find the four stationary points (For your convenience: $(2, 2)$ is one of them.)

d. Find the local behavior around the stationary points. Classify them as node, star point, ... , stable or unstable.
Make a phase portrait, i.e. sketch a few solution curves consistent with your classification.

e. Describe what happens with the populations on the long run. Globally explain how this depends on the initial population sizes.

4. For the (homogeneous) system $\mathbf{x}'(t) = A\mathbf{x}(t) = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}(t)$

the solutions $\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t}$ en $\mathbf{x}_2(t) = \begin{bmatrix} 3+16t \\ -1+16t \end{bmatrix} e^{-3t}$ are given.

a. From the above deduce (so *not* by evaluating the characteristic polynomial!) what are the eigenvalues and corresponding eigenvectors of A.

b. Find the solution of the homogeneous system that at $t = 0$ starts from the point $(6, 2)$. Give explicitly the two component functions $x(t)$ and $y(t)$ of $\mathbf{x}(t)$.

c. Classify the equilibrium point (0,0) and sketch a few solutions in the phase plane. What is the behavior of the solutions $\mathbf{x}(t)$ for $t \to \pm\infty$? in your motivation explain the role of the eigenvector(s).

Now consider the system $\mathbf{x}'(t) = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -2te^{-3t} \\ -2te^{-3t} \end{bmatrix}$.

d. Use variation of parameters to find a particular solution for this (non-homogeneous) system.

5. By the method of separation of variables (no ready-made formulas!) find the solution of the following wave equation with initial values and boundary values:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = 16\dfrac{\partial^2 u}{\partial x^2} & \text{(I)} \\ u_x(0,t) = u_x(4,t) = 0 & \text{(II)} \\ u(x,0) = h(x) & \text{(III-a)} \\ u_t(x,0) = 0 & \text{(III-b)} \end{cases}$$

for $0 \le x \le 4$, en $t \ge 0$.

Uitwerkingen

1a First put $g(t)$ in a suitable (standard) form:
$g(t) = t\cdot(1-u_2(t)) + 2u_2(t) = t-(t-2)u_2(t)$.
The L-transform becomes $G(s) = \dfrac{1}{s^2} - \dfrac{e^{-2s}}{s^2}$.

1b Laplace transform (taking into account $s^2+4s+4=(s+2)^2$):

$$s^2Y(s) - 0s - 2 + 4(sY(s)-0) + 4Y(s) = \frac{2}{(s+2)^3} \;\Leftrightarrow\; Y(s) = \frac{2}{(s+2)^2} + \frac{2}{(s+2)^5}.$$

Inverse transform (noting that $\mathcal{L}^{-1}\left[1/s^5\right] = t^4/4! = \frac{1}{24}t^4$):

$$y(t) = 2te^{-2t} + \tfrac{2}{24}t^3e^{-2t} = 2te^{-2t} + \tfrac{1}{12}t^4e^{-2t}.$$

2a Obviously A has dependent columns, so there exists a $\mathbf{v}\neq\mathbf{0}$ for which $A\mathbf{v}=\mathbf{0}=0\mathbf{v}$. This $\mathbf{v}$ then is an eigenvector for $\lambda=0$.

It's not very hard to find such a $\mathbf{v}$: the first two columns of A are equal, so $\mathbf{v} = \begin{bmatrix}1\\-1\\0\end{bmatrix}$ does the job.

Other sufficient argument: A has dependent columns, so $\text{Det}(A) = \text{Det}(A-0\,I) = 0$, showing that 0 is a root of the characteristic equation $\text{Det}(A-\lambda\,I) = 0$.

2b First a reduction step using the first row, next a reduction step using the first column:

$$\begin{vmatrix} 1-\lambda & 1 & \boxed{1} \\ 1 & 1-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ \lambda & -\lambda & 0 \\ 1 & 1 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2-\lambda & 1 \\ \lambda & 0 & 0 \\ 1 & 2 & -2-\lambda \end{vmatrix}$$

and then cofactor expansion using the second row:

$$\begin{vmatrix} 1-\lambda & 2-\lambda & 1 \\ \lambda & 0 & 0 \\ 1 & 2 & -2-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 2-\lambda & 1 \\ 2 & -2-\lambda \end{vmatrix} = -\lambda\big((2-\lambda)(-2-\lambda)-2\big) = -\lambda(\lambda^2-6)$$

Equating to 0 gives $\lambda_1 = 0,\quad \lambda_{2,3} = \pm\sqrt{6}$.

2c See Lay, § 5.3.

2d The matrix has eigenvalues $\lambda_{1,2}=1$, $\lambda_3=-1$, $\lambda_4=2$. Necessary diagonalizability: two independent eigenvectors for $\lambda_{1,2}$:

$$\begin{bmatrix} 1-1 & 2 & a & 4 \\ 0 & -1-1 & 1 & 0 \\ 0 & 0 & 1-1 & 1 \\ 0 & 0 & 0 & 2-1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & a & 4 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & \boxed{2} & a & 4 \\ 0 & 0 & \boxed{1+a} & 4 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The (possible) pivots are framed. There are 3 pivots if $1+a\neq0$, in which case there's only 1 (indep.) eigenvector for $\lambda_{1,2}$.
Only if $a=-1$ are there 2 pivots, leading to $4-2$ independent eigenvectors.
Conclusion: the matrix is only diagonalizable if $a=-1$.

3a The interaction follows from the coefficients of the terms xy. Both are negative, implying there is mutual competition.

3b Focus on the direction field on the border, consisting of four parts:
For $y = 10,\ 0 < x < 10$ we read off $y'(t) < 0$, so a solution will move into the negative y-direction, i.e. *into* the square. Likewise for the part $x = 10, 0 < y < 10$, $y = 0$, etc.

3c Find solutions of: $\begin{cases} 0.5x\,(6-x-2y) \\ 0.25y\,(8-2x-2y) \end{cases} \longleftrightarrow \begin{cases} x = 0 \text{ of } 6-x-2y = 0 \\ y = 0 \text{ of } 8-2x-2y = 0 \end{cases}$

Combining the four (!) possibilities easily produces the three points $(0,0)$, $(0,4)$ en $(6,0)$, and from $\begin{cases} 6-x-2y = 0 \\ 8-2x-2y = 0 \end{cases}$ follows the fouth point $(2\,2)$.

3d $\begin{cases} \frac{dx}{dt} = 0.5x\,(6-x-2y) \\ \frac{dy}{dt} = 0.25y\,(8-2x-2y) \end{cases} \Longrightarrow \begin{cases} \frac{dx}{dt} = 3x - 0.5x^2 - xy \\ \frac{dy}{dt} = 2y - 0.5xy - 0.5y^2 \end{cases}$

For the liearizations (needed for the local behavior) we need the Jacobian matrix $J(x,y) = \begin{bmatrix} 3-x-y & -x \\ -0.5y & 2-0.5x-y \end{bmatrix}$.

$J(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ has eigenvalues 3 and 2, so (0,0) is an **unstable node**.

$J(6,0) = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}$ has eigenvalues -3, -1, so (6,0) is a **(asymptotically) stable node**.

$J(0,4) = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}$ has eigenvalues -2, -1, which gives again a **stable node**.

$J(2,2) = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$ vraagt iets meer werk.

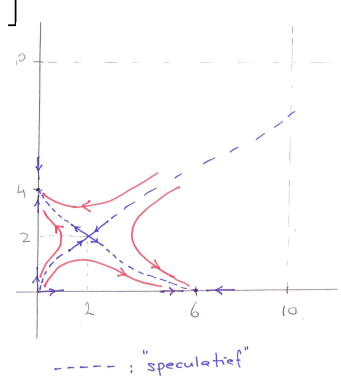
the characteristic polynomial: $\begin{vmatrix} -1-\lambda & -2 \\ -1 & -1-\lambda \end{vmatrix} = (1+\lambda)^2 - 2$ gives the eigenvalues $-1-\sqrt{2} < 0$ and $-1+\sqrt{2} > 0$, so (2,2) is a **sadddle point**.

For a sketch it helps to have the correponsing eigenvectors:

$$\begin{bmatrix} -1-(-1-\sqrt{2}) & -2 \\ -1 & -1-(-1-\sqrt{2}) \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -2 \\ -1 & \sqrt{2} \end{bmatrix} \sim \begin{bmatrix} \sqrt{2} & -2 \\ 0 & 0 \end{bmatrix} \quad \text{e.v.} \quad \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} \approx \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

Likewise $\lambda = -1 + \sqrt{2}$

gives the e.v. $\begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix} \approx \begin{bmatrix} 1.5 \\ -1 \end{bmatrix}$.



3e An initial value close to the point (0,4) will give a solution converging to (0,4) (population I dies out), and analogously around the other stable node (6,0).
Theoretically there will be a solution curve starting at (0,0) and converging to (2,2), and on from 'infinity' to (2,2) (the dotted curve which will at some point on one of the lines $x = 10$ or $y = 10$ will enter the square $[0, 10] \times [0, 10]$). These two curves form the separatrix between the the regions of attraction of the stationary points (0,4) and (6,0).

4a From the solution $\mathbf{x}_1(t)$ it follows that A has the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for the eigenvalue $\lambda = -3$.
From the second solution it follows that -3 is an eigenvalue of multiplicity 2 (and $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is a generalized eigenvector, but that you don't (have to) know).

4b For this we have to find c_1, c_2 for which

$$c_1\mathbf{x}_1(0) + c_2\mathbf{x}_2(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}.$$

One easily finds $c_1 = 3$, $c_2 = 1$, and so the **answer** $\begin{cases} x(t) &= 6e^{-3t} + 16t\,e^{-3t} \\ y(t) &= 2e^{-3t} + 16t\,e^{-3t} \end{cases}$.

4c Because of the repeated negative eigenvalue with only one eigenvector: (0,0) is a stable improper node. Solution curves 'come from' (i.e. from $t = -\infty$) the direction $\pm \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and 'deflect' to approach (0,0) fro the opposite direction, i.e. along $\mp \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
looking at the direction at e.g. the point (0,1): $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$, one may observe that solutions above the line $y = x$ come 'from the north-east' anf turn to the left to approach (0,0) from the third quadrant . (For a more precise illustration: see B&dP § 9.1.)

4d Via the fundamental matrix $F(t) = \begin{bmatrix} 1 & 3+16t \\ 1 & -1+16t \end{bmatrix} e^{-3t}$

with determinant $1 \cdot (-1+16t) - 1 \cdot (3+16t) \cdot (e^{-3t})^2 = -4\,e^{-6t}$

and inverse matrix $\dfrac{1}{-4e^{-6t}} \begin{bmatrix} -1+16t & -(3+16t) \\ -1 & 1 \end{bmatrix} e^{-3t} = \dfrac{1}{4} \begin{bmatrix} 1-16t & 3+16t \\ 1 & -1 \end{bmatrix} e^{3t}$

we find (in a completely standard way) by putting $\mathbf{x}_p(t) = F(t)\mathbf{u}(t)$
that $\mathbf{u}(t)$ has to satisfy $F(t)\mathbf{u}'(t) = \mathbf{g}(t)$, that is (completely standard again)

$$\mathbf{u}'(t) = F^{-1}(t)\mathbf{g}(t) = \frac{1}{4} \begin{bmatrix} 1-16t & 3+16t \\ 1 & -1 \end{bmatrix} e^{3t} \begin{bmatrix} -2t \\ -2t \end{bmatrix} e^{-3t} = \begin{bmatrix} -2t \\ 0 \end{bmatrix}$$

Rthare nice, isn't it! ;-) From this,

$$\mathbf{u}(t) = \begin{bmatrix} -t^2 \\ 0 \end{bmatrix} \implies \mathbf{x}_p(t) = F(t)\mathbf{u}(t) = \ldots = \begin{bmatrix} -t^2e^{-3t} \\ -t^2e^{-3t} \end{bmatrix}$$

Besides, in this exercise it's quicker to just solve the system $F(t)\mathbf{u}'(t) = \mathbf{g}(t)$ 'directly':

$$\left[\begin{array}{cc|c} e^{-3t} & (3+16t)e^{-3t} & -2te^{-3t} \\ e^{-3t} & (-1+16t)e^{-3t} & -2te^{-3t} \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & (3+16t) & -2t \\ 1 & (-1+16t) & -2t \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & (3+16t) & -2t \\ 0 & -4 & 0 \end{array}\right]$$

from which we find: $u_2'(t) = 0, \quad u_1'(t) = -2t, \quad \text{enz.} \; . \; . \; . \; .$

5 The usual three-step appproach:
[I] Put $u(x,t) = X(x)T(t)$, and rewrite to separate separate variables:

$$\frac{T''(t)}{T(t)} = 16\frac{X''(x)}{X(x)} = \text{constant}$$

[II]

$$\left\{ \begin{array}{l} \dfrac{X''(x)}{X(x)} = C \\ X'(0) = 0, \;\; X'(4) = 0 \end{array} \right\} \Rightarrow \ldots \Rightarrow \text{for } C = -\left(\frac{n\pi}{4}\right)^2 : \quad \begin{array}{l} X_n(x) = \cos\left(\frac{n\pi}{4}x\right), \\ \text{for } n = \mathbf{0},\, 1,\, 2,\, \ldots. \end{array}$$

For the above values of C:

$$\frac{T''(t)}{T(t)} = 16C = -(n\pi)^2 \;\; \Rightarrow \;\; \ldots \;\; \Rightarrow \quad T_n(t) = A_n \cos(n\pi t) + B_n \sin(n\pi t)$$

All in all

$$u(x,t) = \sum_{n=0}^{\infty} \Big[A_n \cos(n\pi t) + B_n \sin(n\pi t)\Big] \cos\left(\frac{n\pi}{4}x\right)$$

$u(x,0) = h(x)$ geeft $h(x) = \sum\limits_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{4}x\right)$,
so the A_n are the coefficients of the cosine expansion of $h(x)$.

Written out: $A_0 = \frac{1}{4}\displaystyle\int_0^4 h(x)\,dx$, and $A_n = \frac{2}{4}\displaystyle\int_0^4 h(x)\cos\left(\frac{n\pi}{4}x\right)\,dx$, for $n \geq 1$.

[III] Lastly, $u_t(x,0) = 0$ gives $\displaystyle\sum_{n=1}^{\infty} B_n n\pi \cos\left(\frac{n\pi}{4}x\right) = 0$, from which we conclude that $B_n = 0$.

(This could already have been concluded solving the equation $T''(t)/T(t) = 16C$.)