$n = 1, 2, 3 \ldots$

Technische Universiteit Delft, Fac. EWI

Exam Differential Equations, AESB2110, 3 February 2017, 13:30-16.30

Each answer must be clearly motivated.
You receive a table with Laplace transformations and a few integrals. You may use a simple calculator (which actually you won't need.).
The (maximum) scores: exc.1: **8** pt; exc.2: **8** pt; exc.3: **9** pt; exc.4: **7** pt; exc.5: **10** pt.

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1. **a.** Let $g(t) = \begin{cases} \sin t, & \text{if } 0 \le t \le \pi; \\ 2\sin t, & \text{if } \pi \le t \le 2\pi; \\ 0, & \text{if } t \ge 2\pi. \end{cases}$ Find the Laplace transform of $g(t)$.

b. Find the solution of the initial value problem $\begin{cases} y'' + 5y' + 6y = h(t) \\ y(0) = y'(0) = 0. \end{cases}$

for a 'general' function $h(t)$. (The answer will contain a convolution.)

c. Using part **b.** find the explicit solution (involving no more convolutions) of the initial value problem $\begin{cases} y'' + 5y' + 6y = e^{-3t} \\ y(0) = y'(0) = 0. \end{cases}$

2. **a.** Find the eigenvalues of the following matrix $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

Is this matrix diagonalizable? (Explain!!)

b. True or false: if A and B are two $n \times n$ matrices then (*always*) $\det(AB) = \det(BA)$.

c. True or false:
if A is an $m \times n$ matrix and B an $n \times m$ matrix, then (*always*) $\det(AB) = \det(BA)$.

3. Consider the linear first order system of differential equations $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 8 & -2 \end{bmatrix} \mathbf{x}$

a. Find the general solution of the system.

b. Find the solution passing through the point $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ at $t = 0$.

Write this solution in the form $\begin{cases} x_1(t) = \ldots. \\ x_2(t) = \ldots. \end{cases}$

c. Find a particular solution for the system

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 8 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \cos(2t) + \sin(2t) \\ 4\sin(2t) \end{bmatrix},$$ *using variation of parameters.*

4. Consider the system of differential equations $\begin{cases} x' = x(2-x-y) \\ y' = (1-y)(2-x) \end{cases}$

a. Find all (three) stationary points.

b. Determine the local behaviour of the solutions around these stationary points (i.e. classify them as nodes, spiral points etc.) by considering the linearizations.

c. Give a sketch of the solutions that start from the points $(-2,1)$ and (2,1). Note that at these points $y' = 0$.

5. Consider the following partial differential equation with boundary values:

$$\begin{cases} u_t = \frac{1}{16} u_{xx}, \\ u_x(0,t) = 0, \;\; u_x(10,t) = 0, \;\; \text{for } t \geq 0 \\ u(x,0) = g(x), \;\; \text{for } 0 \leq x \leq 10 \end{cases}$$

in the domain $D\colon 0 \leq x \leq 10, \;\; 0 \leq t$.
Interpretation: u is the temperature of a rod of length 10 that is insolated at the endpoints. (Note that u_x denotes the derivative with respect to x.)

a. Using the method of separation of variables (so no ready made solutions!) find the solution of the above initial value problem.

b. Find the explicit solution in case the function g is given by
$g(x) = \frac{1}{10}\big(\cos(\pi x) + 2\cos(2\pi x)\big)$.

c. Give a set of 'basic' solutions $u_n(x,t)$ if the boundary condition at $x = 0$ is altered to $u(0,t) = 0$,
i.e. non-trivial solutions for $\begin{cases} u_t = \frac{1}{16} u_{xx}, \\ u(0,t) = 0, \;\; u(10,t) = 0, \;\; \text{for } t \geq 0 \end{cases}$