Technical University Delft, Fac. EWI

Exam Differential Equations, AESB2110, 9 November 2016, 13:30 h – 16.30 h

Unless stated otherwise each answer must be clearly motivated.
You may use a simple calculator (which actually you won't need.).
The (maximum) scores: exc.1: **6** pt; exc.2: **7** pt; exc.3: **7** pt; exc.4: **10** pt; exc.5: **10** pt.

2 **1.** **a.** Find the Laplace transform of the function $g(t) = \begin{cases} 0\,, & \text{if } 0 \le t \le 2 \\ 5 - t\,, & \text{if } 2 \le t \le 4 \\ 1\,, & \text{if } t \ge 4 \end{cases}$

4 **b.** Use the Laplace transform to find the solution of the initial value problem
$y''(t) + 4y'(t) + 4y(t) = 2te^{-2t}, \;\; y(0) = 1, \;\; y'(0) = -4.$

1 **2.** **a.** Find the (real) eigenvalues of the matrix $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

4 **b.** Find bases for each eigenvalue and check whether A is diagonalizable.

2 **c.** For this question **check your answer.**
(More than one calculation error means: Zero points!)

Find the value of the determinant of the matrix $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

3. The matrix A is given by $A = \begin{bmatrix} -4 & -3 \\ 6 & 2 \end{bmatrix}$.

5 **a.** Find the general (**real**) solution of the system of differential equations $\mathbf{x}'(t) = A\mathbf{x}(t)$.

2 **b.** Find the solution that passes through the point $(5, 0)$ at time $t = 0$.

Write the solution in the form $\begin{cases} x_1(t) & = \dots \\ x_2(t) & = \dots \end{cases}$

Give a sketch of this solution in the phase plane.

4. Consider the system of differential equations $\begin{cases} x' = (x^2 - 1)y \\ y' = x(2 - y) \end{cases}$

10

$1\frac{1}{2}$ **a.** Find the stationary points.

4 **b.** Determine the local behaviour of the solutions around these stationary points (i.e. classify them as nodes, spiral points etc.) by considering the linearizations.

1 **c.** Find out if there are solution curves $x = \text{constant}$ or $y = \text{constant}$.

$1\frac{1}{2}$ **d.** Sketch a phase portrait that is consistent with the properties found in the first **three** parts of the question.

2 **e.** Find an explit solution for the system starting from the initial value $(0, 2)$.
(Note that this has to do with one of the cases found in part **b**!)
Check (and explain) that this solution is consistent with your phase portrait.

5. Consider the following partial differential equation with boundary values:

$$\begin{cases} u_t = \frac{1}{25} u_{xx}, \\ u_x(0,t) = 0, \;\; u_x(10,t) = 0, \;\; \text{for } t \ge 0 \\ u(x,0) = g(x), \;\; \text{for } 0 \le x \le 10 \end{cases}$$

in the domain $D\colon 0 \le x \le 10, \;\; 0 \le t.$

Interpretation: u is the temperature of a rod of length 10 that is insolated at the endpoints. (Note that u_x denotes the derivative with respect to x.)

10

6 **a.** Using the method of separation of variables (so no ready made solutions!) find the solution of the above initial value problem.

2 **b.** Find the explicit solution in case the function g is given by
$g(x) = \frac{1}{10}\big(\cos(\pi x) - \cos(2\pi x)\big).$

2 **c.** Give a set of 'basic' solutions $u_n(x,t)$ if the boundary condition at $x = 10$ is altered to $u(10,t) = 0$,

i.e. non-trivial solutions for $\begin{cases} u_t = \frac{1}{25} u_{xx}, \\ u_x(0,t) = 0, \;\; u(10,t) = 0, \;\; \text{for } t \ge 0 \end{cases}$

Solutions ½

1a First put $g(t)$ in a suitable (standard) form:
$g(t) = (5-t)\cdot(u_2(t) - u_4(t)) + u_4(t) = 3u_2(t) - (t-2)u_2(t) + (t-4)u_4(t)$.

The L-transform then becomes $G(s) = 3\frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-4s}}{s^2}$. ½

1b Laplace transform (taking into account $s^2+4s+4 = (s+2)^2$):
$s^2Y(s) - s - (-4) + 4(sY(s) - 1) + 4Y(s) = \frac{2}{(s+2)^2} \iff Y(s) = \frac{s}{(s+2)^2} + \frac{2}{(s+2)^4}$. 1½

Now $\frac{s}{(s+2)^2} = \frac{s+2}{(s+2)^2} - \frac{2}{(s+2)^2} = \frac{1}{s+2} - \frac{2}{(s+2)^2} \xrightarrow{\mathcal{L}^{-1}} e^{-2t} - 2te^{-2t}$, 1½

and $\frac{2}{(s+2)^4} = \frac{1}{3}\frac{3!}{(s+2)^4} \to \frac{1}{3}t^3e^{-2t}$, ½ , ½ als foutje en makkelijker bv $\frac{4}{(s+2)^5}$

which gives the final answer $y(t) = e^{-2t} - 2te^{-2t} + \frac{1}{3}t^3e^{-2t}$ ½

2a $\mathrm{Det}(A - \lambda I) = \ldots = (2-\lambda)(1-\lambda)(2-\lambda)(1-\lambda) = (\lambda-2)^2(\lambda-1)^2$ so A has two eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = 1$, both with algebraic multiplicity 2.

1

2b To find the eigenvectors for $\lambda = 2$:

the null space of $(A - 2I) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has dimension 2, so ½

eigenvalue 2 has algebraic multiplicity 2

and geometric multiplicity 2. A (possible) basis of eigenvectors: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ 1

Likewise, $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}$ is a basis of eigenvectors for $\lambda = 1$. 1½

The two bases put together give a basis of eigenvectors for $\mathbb{R}^4$, so A is diagonalizable. 1

2c

$$\begin{vmatrix} 2 & \boxed{1} & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -3 & 2 & 1 & 0 \\ -2 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = (-1)\begin{vmatrix} -3 & 1 & 0 \\ -2 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \ldots = 5$$ 0, 1 or 2

3a First: eigenvalues and eigenvectors of A:

$$\operatorname{Det}(A-\lambda I)=\begin{vmatrix}-4-\lambda & -3\\ 6 & 2-\lambda\end{vmatrix}=(-4-\lambda)(2-\lambda)-(-3)\cdot 6=\lambda^2+2\lambda+10=(\lambda+1)^2+9$$

Has zeros (= eigenvalues) $\lambda_{1,2}=-1\pm 3i$. $1\frac{1}{2}$

The eigenvectors for $\lambda=-1+3i$:

the null space of $A-(-1+3i)I=\begin{bmatrix}-3-3i & -3\\ 6 & 3-3i\end{bmatrix}\sim\begin{bmatrix}1+i & 1\\ 2 & 1-i\end{bmatrix}\sim\begin{bmatrix}1+i & 1\\ 0 & 0\end{bmatrix}$:

An eigenvector: $\mathbf{v}=\begin{bmatrix}-1\\ 1+i\end{bmatrix}$. $1\frac{1}{2}$

The **complex** solution

$$\begin{aligned}\mathbf{v}e^{(-1+3i)t} &= \begin{bmatrix}-1\\ 1+i\end{bmatrix}e^{-t}(\cos 3t+i\sin 3t)\\ &= e^{-t}\begin{bmatrix}-\cos 3t\\ \cos 3t-\sin 3t\end{bmatrix}+ie^{-t}\begin{bmatrix}-\sin 3t\\ \cos 3t+\sin 3t\end{bmatrix}\end{aligned}$$ $1\frac{1}{2}$

leads to the **real** solution

$$\mathbf{x}(t)=C_1e^{-t}\begin{bmatrix}-\cos 3t\\ \cos 3t-\sin 3t\end{bmatrix}+C_2e^{-t}\begin{bmatrix}-\sin 3t\\ \cos 3t+\sin 3t\end{bmatrix}$$ $\frac{1}{2}$

3b To get a solution for which $\mathbf{x}(0)=\begin{bmatrix}5\\ 5\end{bmatrix}$,

we have to solve $C_1\begin{bmatrix}-1\\ 1\end{bmatrix}+C_2\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}5\\ 0\end{bmatrix}$ for C_1, C_2.

This is easily done and gives $C_1=-5$, $C_2=5$, so that the answer becomes

$$\begin{aligned}\mathbf{x}(t) &= (-5)\,\mathbf{x}_1(t)+5\,\mathbf{x}_2(t)\\ &= \ldots = 5e^{-t}\begin{bmatrix}\cos 3t-\sin 3t\\ 2\sin 3t\end{bmatrix}\end{aligned}$$ $1/2$

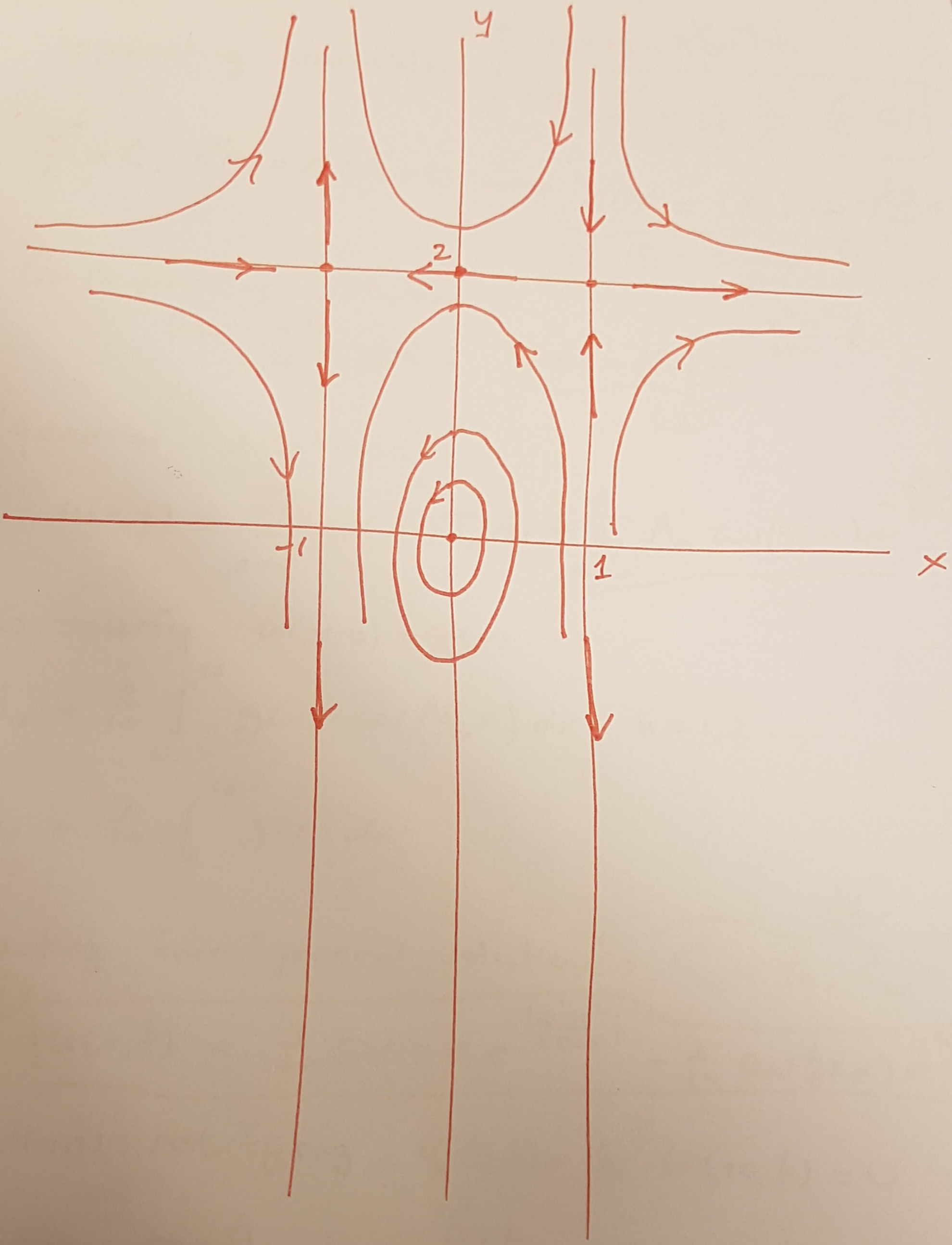
That is: $\begin{cases}x_1(t) = 5e^{-t}(\cos 3t-\sin 3t)\\ x_2(t) = 10e^{-t}\sin 3t.\end{cases}$ $1/2$

+ picture: 1

The origin is a stable spiral point,

and since at the point $\mathbf{x}=\begin{bmatrix}1\\ 0\end{bmatrix}$ the derivative (= direction of the solution curve) is given by

$\mathbf{x}'=A\mathbf{x}=\begin{bmatrix}-4\\ 6\end{bmatrix}$, the solution spirals around the origin counter clockwise.


y
2
-1
1
x

1/2

4a We have to solve $\begin{cases} (x^2-1)y = 0 \\ x(2-y) \ = 0 \end{cases}$ $x(2-y)=0 \Leftrightarrow x=0$ or $y=2$.
For $x=0$, the first equation gives $y=0$, and for $y=2$ you get the correspond values $x=\pm 1$.
So there are three stationary points: $(0,0)$, $(-1,2)$ and $(1,2)$.

4b The Jacobian matrix: $J(x,y) = \begin{bmatrix} 2xy & x^2-1 \\ 2-y & -x \end{bmatrix}$. 1

$J(0,0) = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$ has characteristic polynomial λ^2+2, which gives eigenvalues $\pm i\sqrt{2}$.
So the linearized system has a center at the origin, and for the non-linear system (0,0) is either a center or a (stable or unstable) spiral point. 1

$J(-1,2) = \begin{bmatrix} -4 & 0 \\ 0 & 1 \end{bmatrix}$, a diagonal matrix with eigenvalues -4 and 1, so $(-1,2)$ is a saddle point.
$J(1,2) = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$, a diagonal matrix with eigenvalues 4 and -1, so $(1,2)$ is also a saddle point. 1

Furthermore, given the simple form of the last two matrices it is clear that for the linearization around (−1,2), an eigenvector for $\lambda=-4$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and an eigenvector for $\lambda=1$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Likewise, around (1,2), the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ belongs to the positive eigenvalue and the vector $\begin{bmatrix} -0 \\ 1 \end{bmatrix}$ to the negative eigenvalue. 1

1 **4c** $x=-1$, $x=1$, $y=2$

4d $1\frac{1}{2}$

4e If $y=2$, then $y'=0$, so y is constant, namely $y=2$.
The corresponding DE for x: $x' = (x^2-1)\cdot 2$, a **separable** differential equation.
I separate the variables and integrate: 1/2

$$\frac{1}{x^2-1}\frac{dx}{dt} = 2 \Rightarrow \int \frac{1}{x^2-1}\,dx = \int \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}\,dx = \int 2\,dt \Rightarrow$$

$$\Rightarrow \ln\left|\frac{x-1}{x+1}\right| = 2t + C.$$ 1

Plugging in the initial value $x(0)=0$ gives $C = \ln 1 = 0$.
Lastly: rewrite the implicit solution $\ln\left|\frac{x-1}{x+1}\right| = 2t$. Note that since we start with x between -1 and 1, the absolute value of $\dfrac{x-1}{x+1}$ equals $\dfrac{1-x}{x+1}$

$$\left|\frac{x-1}{x+1}\right| = e^{2t} \Leftrightarrow \frac{1-x}{x+1} = e^{2t} \Leftrightarrow 1-x = (1+x)e^{2t} = e^{2t} + e^{2t}x \Leftrightarrow$$

$$\Leftrightarrow (e^{2t}+1)x = 1 - e^{2t} \Leftrightarrow x = \frac{1-e^{2t}}{e^{2t}+1}.$$ 1/2

So x takes on values between 0 and -1, and if $t\to\infty$, then $x\to -1$, which means that the solution curve converges to the stationary point $(-1,2)$, which is consistent with the saddle point we found there earlier. 1

(= BONUS POINT)

[5] ~~(short)~~ solution

separating variables: $\begin{cases} u(x,t) = X(x)T(t) \\ \rightarrow \boxed{\frac{T'}{T} = \frac{1}{25}\frac{X''}{X} \; (= c)} \end{cases}$ (1)

$\frac{X''}{X} = c,\ X'(0) = X'(10) = 0 \longrightarrow \underbrace{X_n(x) = (A_n)\cos\left(\frac{n\pi}{10}x\right)}$

(2) (for complete derivation)

Corresponding solution for $\frac{T'}{T} = \frac{1}{25}c$: $\underbrace{T_n(t) = (C_n)\,e^{-\frac{n^2\pi^2}{2500}\cdot t}}$

(1)

"general" solution:

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x)T_n(t) = \underbrace{\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{10}x\right)e^{-\frac{n^2\pi^2}{2500}t}}$$

(1)

To satisfy $u(x,0) = g(x)$:

(1) $\begin{cases} A_n = \frac{2}{10}\int_0^{10} g(x)\cdot\cos\left(\frac{n\pi}{10}x\right)dx, & n = 1, 2, \ldots \\ A_0 = \frac{1}{10}\int_0^{10} g(x)\,dx \end{cases}$

(1)

b] starting from "general solution": $\overbrace{C_{10} = \frac{1}{10},\ C_{20} = \frac{-1}{10}}$

(1) so $\boxed{u(x,t) = \frac{1}{10}\cos(\pi x)\,e^{-\frac{10^2\pi^2}{2500}t} - \frac{1}{10}\cos(2\pi x)\,e^{-\frac{20^2\pi^2}{2500}t}}$

c] solutions satisfying $u_x(0,t) = 0,\ u(10,t) = 0$:

$$u_n(x,t) = \cos(\lambda_n x)\cdot e^{-\frac{\lambda_n^2}{25}t}, \quad \lambda_n = \frac{n + \frac{1}{2}\pi}{10}$$ (2)