Analyse 4 augustus 2013 Wi 1300TA

1. a) $\underline{r}(t) = \begin{pmatrix} 2t \\ t^2 \end{pmatrix}, \quad 0 \le t \le 1.$

$$\underline{r}'(t) = \begin{pmatrix} 2 \\ 2t \end{pmatrix}, \quad |\underline{r}'(t)| = \sqrt{4+4t^2} = 2\sqrt{1+t^2}$$

$$m = \int_C \rho\, ds = \rho \int_0^1 |\underline{r}'(t)|\, dt = \rho \int_0^1 2\sqrt{1+t^2}\, dt \underset{\uparrow}{=}$$

formuleblad

$$\rho \left[t\sqrt{1+t^2} + \ln\left(t + \sqrt{1+t^2}\right) \right]_0^1 =$$

$$\rho \left(\sqrt{2} + \ln\left(1+\sqrt{2}\right) \right).$$

b) $$\bar{x} = \frac{1}{m} \int_C \rho\, x\, ds = \frac{1}{m} \int_0^1 \rho \cdot 2t \cdot 2\sqrt{1+t^2}\, dt =$$

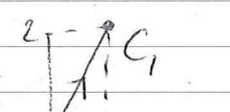
$$\frac{\rho}{m} \int_0^1 4t\sqrt{1+t^2}\, dt = \frac{\rho}{m} \left[\frac{4}{3} (1+t^2)^{3/2} \right]_0^1 =$$

$$\frac{\rho}{m} \frac{4}{3} (2\sqrt{2} - 1) = \frac{1}{\cancel{\rho}(\sqrt{2} + \ln(1+\sqrt{2}))} \cdot \frac{4}{3} (2\sqrt{2} - 1).$$

2. $$\underline{F}(x,y) = \begin{pmatrix} ye^x + y^2 + y \\ e^x + 2xy - x \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix}$$

$$\frac{\partial P}{\partial y} = e^x + 2y + 1, \quad \frac{\partial Q}{\partial x} = e^x + 2y - 1$$

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ dus $\underline{F}$ is niet conservatief.

a) 

$$\underline{r}(t) = \begin{pmatrix} t \\ 2t \end{pmatrix}, \quad 0 \le t \le 1, \quad \underline{r}'(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\int_{C_1} \underline{F} \cdot d\underline{r} = \int_0^1 \underline{F} \cdot \underline{r}'(t)\, dt = \int_0^1 \begin{pmatrix} 2te^t + 4t^2 + 2t \\ e^t + 4t^2 - t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} dt$$

$$= \int_0^1 (2te^t + 4t^2 + 2t + 2e^t + 8t^2 - 2t)\, dt = \left[2te^t + 4t^3 \right]_0^1 = 2e + 4.$$

2.b) C_2 is een gesloten kromme, negatief georiënteerd,
D is het gebied binnen C_2.

Volgens de stelling van Green geldt nu:

$$\int_{C_2} \underline{F} \cdot d\underline{r} = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA =$$

$$-\iint_D (-2)\, dA = 2 \iint_D dA = 2\, A(D) = 2 \cdot \pi \cdot 9$$
$$= 18\pi$$

3.a). Er zijn 2 mogelijkheden:

1) Laat zien dat Curl $\underline{G} = \underline{0}$

2) Bepaal een potentiaalfunctie $f(x,y,z)$

1). $\text{curl}\, \underline{G} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} 4x + yz \\ xz \\ \frac{2z}{1+z^2} + xy \end{pmatrix} = \begin{pmatrix} x - x \\ -(y-y) \\ z - z \end{pmatrix} = \underline{0}$

2) $\frac{\partial f}{\partial x} = 4x + yz \rightarrow f = 2x^2 + xyz + g(y,z)$.

$$\frac{\partial f}{\partial y} = xz + g_y(y,z) = xz$$

Dus $g_y(y,z) = 0 \rightarrow g(y,z) = h(z)$

$f = 2x^2 + xyz + h(z)$

$f_z = xy + h'(z) = \frac{2z}{1+z^2} + xy \rightarrow h'(z) = \frac{2z}{1+z^2}$

$h(z) = \ln(1+z^2)$

$f(x,y,z) = 2x^2 + xyz + \ln(1+z^2)$ is een potentiaalfunctie van $\underline{G}$

b) Als a) aangetoond is met curl $\underline{G} = \underline{0}$ kan je nu het beste alsnog een potentiaalfunctie bepalen.
Vervolgens:

$$\int_K \underline{G} \cdot d\underline{r} = f(\underline{r}(2)) - f(\underline{r}(1))$$
$$= f(3, \sqrt{2}, 2) - f(-1, 0, 0)$$
$$= 18 + 6\sqrt{2} + \ln 5 - 2$$
$$= 16 + 6\sqrt{2} + \ln 5.$$

4 Maak een parametrisering (ik geef hier 2 mogelijkheden)

1) $\underline{r}(r,t) = \begin{pmatrix} r^2 \\ r\cos t \\ r\sin t \end{pmatrix}, \quad 0 \le r \le \sqrt{2}, \quad 0 \le t \le 2\pi.$

$$\underline{r}_r \times \underline{r}_t = \begin{pmatrix} 2r \\ \cos t \\ \sin t \end{pmatrix} \times \begin{pmatrix} 0 \\ -r\sin t \\ r\cos t \end{pmatrix} = \begin{pmatrix} r \\ -2r^2\cos t \\ -2r^2\sin t \end{pmatrix}$$

$$|\underline{r}_r \times \underline{r}_t| = r\sqrt{1 + 4r^2\cos^2 t + 4r^2\sin^2 t} = r\sqrt{1+4r^2}$$

$$A(S) = \iint_S dS = \int_{t=0}^{2\pi} \int_{r=0}^{\sqrt{2}} r\sqrt{1+4r^2}\, dr\, dt =$$

$$2\pi \left[\frac{1}{8}\frac{2}{3}(1+4r^2)^{3/2}\right]_0^{\sqrt{2}} = \frac{\pi}{6}(27-1) = \frac{13}{3}\pi$$

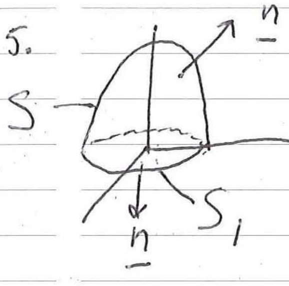
2) $\underline{r}(y,z) = \begin{pmatrix} y^2+z^2 \\ y \\ z \end{pmatrix}, \quad y^2+z^2 \le 2$

$$|\underline{r}_y \times \underline{r}_z| = \left|\begin{pmatrix} 2y \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2z \\ 0 \\ 1 \end{pmatrix}\right| = \left|\begin{pmatrix} 1 \\ -2y \\ -2z \end{pmatrix}\right| = \sqrt{1+4y^2+4z^2}$$

$$A(S) = \iint_{y^2+z^2 \le 2} \sqrt{1+4y^2+4z^2}\, dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{2}} \sqrt{1+4r^2}\, r\, dr\, d\theta =$$

4. vervolg. $A(S) = 2\pi \left[\frac{1}{8} \frac{2}{3} (1+4r^2)^{3/2} \right]_0^{\sqrt{2}} = \frac{13}{3}\pi$.

5.



Sluit het oppervlak S af met S_1.

$S_1 = \{(x,y,z) \mid z = 0,\ x^2+y^2 \le 2\}$

met eenheidsnormaal $\underline{n} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

Bij de divergentiestelling moet het oppervlak dat E omvat georiënteerd zijn volgens naar buiten gerichte eenheidsnormaal $\underline{n}$. (Dat klop nu dus).

$E = \{(x,y,z) \mid 0 \le z \le 2-x^2-y^2,\ x^2+y^2 \le 2\}$.

De divergentiestelling geeft nu:

$$\iint_S \underline{F} \cdot d\underline{S} + \iint_{S_1} \underline{F} \cdot d\underline{S} = \iiint_E \operatorname{div} \underline{F}\, dV$$

Dus $$\iint_{S} \underline{F} \cdot d\underline{S} = \iiint_E \operatorname{div} \underline{F}\, dV - \iint_{S_1} \underline{F} \cdot d\underline{S}.$$

$$\iiint_E \operatorname{div} \underline{F}\, dV = \iiint_E (3x^2+3y^2)\, dV \overset{\text{cilindercoordinaten.}}{=}$$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{2}} \int_{z=0}^{2-r^2} 3r^2 \cdot r\, dz\, dr\, d\theta = 2\pi \int_0^{\sqrt{2}} 3r^3(2-r^2)\, dr =$$

$$2\pi \int_0^{\sqrt{2}} (6r^3 - 3r^5)\, dr = 2\pi \left[\frac{3}{2} r^4 - \frac{1}{2} r^6 \right]_0^{\sqrt{2}} = 2\pi(6-4) = 4\pi.$$

$$\iint_{S_1} \underline{F} \cdot d\underline{S} = \iint_{S_1} \underline{F} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dS = \iint_{S_1} \begin{pmatrix} - \\ - \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dS = -2A(S) = -4\pi$$

Dus $$\iint_S \underline{F} \cdot d\underline{S} = 4\pi - (-4\pi) = 8\pi$$