1.1 $m \times n$ matrix $\rightarrow$ m rijen en n kolommen

1.2 Als er een strijdige vergelijking in de matrix staat dan is er geen oplossing

Als er vrije variabele zijn dan zijn er ∞ veel oplossingen

Anders is er één oplossing.

1.3 Algebraic Properties of $\mathbb{R}^n$

For all u, v, w in $\mathbb{R}^n$ and all scalars c and d:

I : $u + v = v + u$
II : $(u+v) + w = u + (v+w)$
III : $u + 0 = 0 + u = u$
IV : $u + (-u) = -u + u = 0$
V : $c(u+v) = cu + cv$
VI : $(c+d)u = cu + du$
VII : $c(du) = (cd)u$
VIII : $1u = u$

Defention: If $v_1, \dots, v_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $v_1, \dots, v_p$ is denoted by:

$$\text{Span}\{v_1, \dots, v_p\}$$

and is called the subset of $\mathbb{R}^n$ spanned by $v_1, \dots, v_p$ That is, the $\text{Span}\{v_1, \dots, v_p\}$ is the collection of all vectors that can be written in the form

$$c_1 v_1 + \dots + c_p v_p$$

with $c_1, \dots, c_p$ scalars.

Example: let $a_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, and $b = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$. Then $\text{Span}\{a_1$ is a plane through the origin in $\mathbb{R}^3$. Is b in that plane

Solution: $\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

$0 \neq -2$ so b is not in $\text{Span}\{a_1, a_2\}$

1.4 Defenition

If A is an $m \times n$ matrix, with columns $a_1, \dots, a_n$, and if x is in $\mathbb{R}^n$, then the product of A and x, denoted Ax, is the linear combination of the columns of A using the corresponding entries in x as weights; that is,

$$Ax = [a_1 \; a_2 \; a_3 \dots a_n] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$$

Theorem 3
p. 58

If A is an $m \times n$ matrix with columns $a_1, \dots, a_n$, and if b is in $\mathbb{R}^m$, the matrix equation

$$Ax = b$$

has the same solution set as the vector equation

$$x_1 a_1 + x_2 a_2 + \dots + x_n a_n = b$$

Which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[a_1 \; a_2 \dots a_n \; b]$$

Theorem 4
p. 59

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all ~~th~~ treue statements or they are all false.

a. For each b in $\mathbb{R}^m$, the equation $Ax = b$ has a solution.

b. ~~Each~~ Each b in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$

d. A has a pivot position in every row.

Theorem 5
p. 61

If A is an $m \times n$ matrix, u and v are vectors in $\mathbb{R}^n$, and c is a scalar, then:

a. $A(u+v) = Au + Av$

b. $A(cu) = c(Au)$

~~VS The homogeneous equation $Ax=0$ has a nontrivial solution if and only if the~~

1.7 **Definition** p. 81

An indexed set of vectors $\{v_1, \dots, v_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \dots + x_p v_p = 0$$

has only the trivial solution. The set $\{v_1, \dots, v_p\}$ is said to be linearly dependent if there exists weights $c_1, \dots, c_p$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p = 0$$

The columns of a matrix A are linearly independent if and only if the equation $Ax = 0$ has <u>only</u> the trivial solution.

Example 2 p. 82-83

Determine if the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.

Solution: To study $Ax=0$, row reduce the augmented matrix:

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

So $Ax=0$ has only the trivial solution, and the columns of A are linearly independent.

A set of two vectors $\{v_1, v_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Theorem 7 p. 84

Characterization of Linearly Dependent Sets

An indexed set $S = \{v_1, \dots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others

Theorem 8 p. 85

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{v_1, \dots, v_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$

$$\begin{array}{cc} \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} & n \\ p & \end{array}$$

Theorem 9 p.85

If a set $S = \{v_1, \dots, v_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the ~~system~~ set is linearly dependent.

1.8 Defenition A transformation T is linear if:

i. $T(u+v) = T(u) + T(v)$ for all u, v in the domain of T

ii. $T(cu) = cT(u)$ for all u and all scalars c

If T is a linear transformation, then

$$T(0) = 0$$

and

$$T(cu + dv) = cT(u) + dT(v)$$

for all vectors u, v in the domain of T and all scalars c, d.

Example: Given a scalar r, define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = rx$. T is called a contraction when $0 \leq r \leq 1$ and a dilation when $r > 1$. Let $r = 3$, and show that T is a linear transformation.

Solution: Let u, v be in $\mathbb{R}^2$ and let c, d be scalars. Then

$$\begin{aligned} T(cu + dv) &= 3(cu + dv) \\ &= 3cu + 3dv \\ &= c(3u) + d(3v) \\ &= cT(u) + dT(v) \end{aligned}$$

Thus T is a linear transformation.

1.9 The Matrix of a Linear Transformation.

Theorem 10. p. 99 Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(x) = Ax \quad \text{for all } x \in \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose j-th column is the vector $T(e_j)$, where e_j is the j-th column of the identity matrix in $\mathbb{R}^n$:

$$A = [T(e_1) \ \cdots \ T(e_n)]$$

Example: Find the standard matrix A for the trans: $T(x) = 3x$.

Write $T(e_1) = 3e_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $T(e_2) = 3e_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

* 101 - 102 Belangrijke transformatie tabellen!

Defenition p. 103 — A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each b in $\mathbb{R}^m$ is the image of ~~at~~ at least one x in $\mathbb{R}^n$

Defenition p. 103 — A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each b in $\mathbb{R}^m$ is the image of at most one x in $\mathbb{R}^n$

Theorem 11 p. 104 — Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(x) = 0$ has only the trivial solution.

Theorem 12 p. 105 — Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

b. T is one-to-one if and only if the columns of A are linearly independent.

2.1 Matrix ~~mult~~ Operations

Theorem 1 p. 124 — Let A, B, and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

Defenition — If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $b_1, \ldots, b_p$ then the product AB is the $m \times p$ matrix whose columns are $Ab_1, \ldots, Ab_p$. That is,

$$AB = A[b_1 \; b_2 \; \ldots \; b_p] = [Ab_1 \; Ab_2 \; \ldots \; Ab_p]$$

Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

Theorem 2
p 129

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$

b. $A(B+C) = AB + AC$

c. $(B+C)A = BA + CA$

d. $r(AB) = (rA)B = A(rB)$

e. $I_m A = A = A I_n$

The transpose of a matrix

Example:
p. 130

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix} \longrightarrow B^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix} \longrightarrow C^T = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}$$

Theorem 3
p 131

Let A and B ~~be~~ denote matrices whose sizes are appropriate for the following sums and products

a. $(A^T)^T = A$

b. $(A+B)^T = A^T + B^T$

c. For any scalar~~s~~ r, $(rA)^T = rA^T$

d. $(AB)^T = A^T B^T$

The Inverse of a Matrix

Theorem 4
p. 135

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

$\det A = ad - bc$ is called the determinant of A.

Example:
p. 136

$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ $\quad$ $\det A = 3 \cdot 6 - 4 \cdot 5 = -2 \neq 0$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 6/-2 & -4/-2 \\ -5/-2 & 3/-2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

Theorem 5
p. 136

If A is invertible $n \times n$ matrix, then for each b in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$.

Theorem 6
p 137

a. If A is invertible, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB and the inverse of AB is the product of the inverses of A and B in reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

! **Theorem 7**
p. 139

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}

* A aanvullen met I_n en A vegen tot er de eenheidsmatrix staat.

2.8 Subspaces of $\mathbb{R}^n$

Defenition p. 184 — A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties

a. The zero vector is in H

b. For each u and v in H, the sum $u + v$ is in H

c. For each u in H and each scalar c the vector cu is in H.

Defenition p. 185 — The column space of a matrix A is the set $\text{Col}(A)$ of all linear combinations of the columns of A.

Defenition p. 185 — The null space of a matrix A is the set $\text{Nul}(A)$ of all solutions to the homogeneous equation $Ax = 0$

Theorem 12 p. 186 — The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $Ax = 0$ of m homogeneous equations in n unknowns is a subspace of $\mathbb{R}^n$

Defenition p. 186 — A basis for a subspace H of $\mathbb{R}^n$ is a lineard independent set in H that spans H.

! Theorem 13 p. 188 — The pivot columns of a matrix A form a basis for the column space of A.

2.9 Dimension and Rank

Defenition p. 192

Suppose the set $\beta = \{b_1, \ldots, b_p\}$ is a basis for a subspace H. For each x in H, the coördinates of x relative to the basis β are the weights $c_1, \ldots, c_p$ such that $x = c_1 b_1 + \ldots + c_p b_p$ and the vectors in $\mathbb{R}^p$

$$[x]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is called the coördinate vector of x or the β-coördinate vector of x.

Defenition p. 193

The dimension of a nonzero subspace H, denoted by $\dim(H)$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{0\}$ is defined to be zero

Defenition p. 194

The rank of a matrix A, denoted by rank A, is the dimension of the column space of A.

! **Theorem 14** p. 194

If a matrix A has n columns, then ~~the~~ $\text{rank}(A) + \dim \text{Nul}(A) = n$

! **Theorem 15** p. 195

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H

! **Theorem** p. 195

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$

n. $\text{Col}(A) = \mathbb{R}^n$

o. $\dim \text{Col}(A) = n$

p. $\text{Rank}(A) = n$

q. $\text{Nul}(A) = \{0\}$

r. $\dim \text{Nul}(A) = 0$

6.1

Theorem 1 p. 392

Let $u, v,$ and w be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a. $u \cdot v = v \cdot u$

b. $(u+v) \cdot w = u \cdot w + v \cdot w$

c. $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$

d. $u \cdot u \geq 0$, and $u \cdot u = 0$ if and only if $u = 0$

Defenition

The length (or norm) of v is the nonnegative scalar $\|v\|$ defined by

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} \quad \text{and} \quad \|v\|^2 = v \cdot v$$

Example 2 p. 393

Let $v = (1, -2, 2, 0)$. Find a <u>unit</u> vector u in the same direction as v.

Solution: First compute $\|v\|$:

$$\|v\|^2 = v \cdot v = 1^2 + (-2)^2 + 2^2 + 0^2 = 9$$

$$\|v\| = \sqrt{9} = 3$$

Then multiply v by $\frac{1}{\|v\|}$ to obtain

$$u = \frac{1}{\|v\|} v = \tfrac{1}{3} v = \tfrac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

Defenition p. 394

For u and v in $\mathbb{R}^n$, the distance between u and v, written as $\text{dist}(u, v)$, is the length of the vector $u - v$. That is

$$\text{dist}(u, v) = \|u - v\|$$

Defenition p. 395

Two vectors u and v in $\mathbb{R}^n$ are orthogonal if $u \cdot v = 0$

Theorem 2 p. 396

The Pythagorean Theorem

Two vectors u and v are orthogonal if and only if $\|u+v\|^2$ if and only if $\|u+v\|^2 = \|u\|^2 + \|v|$

Theorem 3 p. 397

Let A be a $m \times n$ matrix. The orthogonal complement of the row space of A is the nullspace of A, and the orthogonal complement of the column space of A is the nullspace of A^T:

$$\text{Row}(A)^{\perp} = \text{Nul}(A) \quad \text{and} \quad \text{Col}(A)^{\perp} = \text{Nul}(A^T)$$

6.3 Orthogonal Projections

Theorem 8 p. 411 — The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$. Then each y in $\mathbb{R}^n$ can be written uniquely in the form

$$y = \hat{y} + z$$

Where $\hat{y}$ is in W and z is in $W^{\perp}$. In fact, if $\{u_1, \ldots, u_p\}$ is any orthogonal basis of W, then

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \ldots + \frac{y \cdot u_p}{u_p \cdot u_p} u_p$$

And $z = y - \hat{y}$

Theorem 9 p. 414 — The best Approximation Theorem

Let W be a subspace of $\mathbb{R}^n$, y any vector in $\mathbb{R}^n$, and $\hat{y}$ the orthogonal projection of y onto W. Then $\hat{y}$ is the closest point in W to y, in the sense that

$$\|y - \hat{y}\| < \|y - v\|$$

for all v in W distinct from $\hat{y}$

Theorem 10 p. 415

If $\{u_1, \ldots, u_p\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then

$$\text{proj}_W y = (y \cdot u_1)u_1 + (y \cdot u_2)u_2 + \ldots + (y \cdot u_p)u_p$$

If $U = [u_1 \; u_2 \; \ldots \; u_p]$, then

$$\text{proj}_W y = UU^T y \text{ for all } y \text{ in } \mathbb{R}^n$$

6.4 The Gram-Schmidt Process

! Theorem 11 p. 420 — The Gram-schmidt Process

Given a basis $\{x_1, \ldots, x_p\}$ for a subspace W of $\mathbb{R}^n$, define

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} \cdot v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$\vdots$$

$$v_p = x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \ldots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}}$$

Then $\{v_1, \ldots, v_p\}$ is an orthogonal basis for W. In addition

$$\text{Span}\{v_1, \ldots, v_p\} = \text{Span}\{x_1, \ldots, x_k\} \text{ for } 1 \leq k \leq p$$

Theorem 12 p. 421 — The QR Factorization

If A is an mxn matrix with linearly independent columns, then A can be factored as A = QR, where Q is an mxn matrix whose columns form an ~~orthogonal~~ orthonormal basis for Col(A) and R is an nxn upper triangular invertible matrix with positive entries on its diagonal.

6.5

Defenition p. 425

If A is mxn and b is in $\mathbb{R}^m$, a least-squares solution of $Ax = b$ is an $\hat{x}$ in $\mathbb{R}^n$ such that

$$\|b - A\hat{x}\| \leq \|b - Ax\|$$

for all x in $\mathbb{R}^n$

! Theorem 13 p. 427

The set of least-squares solutions of $Ax = b$ coincides with the nonempty set of solutions of the normal equations $A^TAx = A^Tb$

Theorem 14 p. 429

The matrix A^TA is invertible if and only if the columns of A are linearly independent. In this case, the equation $Ax = b$ has only one least-squares solution $\hat{x}$ and is given by

$$\hat{x} = (A^TA)^{-1}A^Tb$$

Theorem 15 Given an $m \times n$ matrix A with linearly independent columns, let $A = QR$ be a QR factorization of A as in theorem 12. Then, for each b in $\mathbb{R}^m$, the equation $Ax = b$ has a unique least-squares solution, given by

$$\hat{x} = R^{-1} Q^T b.$$