QUANTUM MECHANICS

40

40.1. **IDENTIFY** and **SET UP:** The energy levels for a particle in a box are given by $E_n = \dfrac{n^2h^2}{8mL^2}$.

EXECUTE: **(a)** The lowest level is for $n = 1$, and $E_1 = \dfrac{(1)(6.626\times10^{-34}\ \text{J}\cdot\text{s})^2}{8(0.20\ \text{kg})(1.5\ \text{m})^2} = 1.2\times10^{-67}\ \text{J}$.

(b) $E = \frac{1}{2}mv^2$ so $v = \sqrt{\dfrac{2E}{m}} = \sqrt{\dfrac{2(1.2\times10^{-67}\ \text{J})}{0.20\ \text{kg}}} = 1.1\times10^{-33}\ \text{m/s}$. If the ball has this speed the time it would take it to travel from one side of the table to the other is $t = \dfrac{1.5\ \text{m}}{1.1\times10^{-33}\ \text{m/s}} = 1.4\times10^{33}\ \text{s}$.

(c) $E_1 = \dfrac{h^2}{8mL^2}$, $E_2 = 4E_1$, so $\Delta E = E_2 - E_1 = 3E_1 = 3(1.2\times10^{-67}\ \text{J}) = 3.6\times10^{-67}\ \text{J}$

(d) EVALUATE: No, quantum mechanical effects are not important for the game of billiards. The discrete, quantized nature of the energy levels is completely unobservable.

40.2. $L = \dfrac{h}{\sqrt{8mE_1}}$

$$L = \frac{(6.626\times10^{-34}\ \text{J}\cdot\text{s})}{\sqrt{8(1.673\times10^{-27}\ \text{kg})(5.0\times10^{6}\ \text{eV})(1.602\times10^{-19}\ \text{J/eV})}} = 6.4\times10^{-15}\ \text{m}.$$

40.3. **IDENTIFY:** An electron in the lowest energy state in this box must have the same energy as it would in the ground state of hydrogen.

SET UP: The energy of the n^{th} level of an electron in a box is $E_n = \dfrac{nh^2}{8mL^2}$.

EXECUTE: An electron in the ground state of hydrogen has an energy of $-13.6\ \text{eV}$, so find the width corresponding to an energy of $E_1 = 13.6\ \text{eV}$. Solving for L gives

$$L = \frac{h}{\sqrt{8mE_1}} = \frac{(6.626\times10^{-34}\ \text{J}\cdot\text{s})}{\sqrt{8(9.11\times10^{-31}\ \text{kg})(13.6\ \text{eV})(1.602\times10^{-19}\ \text{J/eV})}} = 1.66\times10^{-10}\ \text{m}.$$

EVALUATE: This width is of the same order of magnitude as the diameter of a Bohr atom with the electron in the K shell.

40.4. **(a)** The energy of the given photon is

$$E = hf = h\frac{c}{\lambda} = (6.63\times10^{-34}\ \text{J}\cdot\text{s})\frac{(3.00\times10^{3}\ \text{m/s})}{(122\times10^{-9}\ \text{m})} = 1.63\times10^{-18}\ \text{J}.$$

The energy levels of a particle in a box are given by Eq.40.9

$\Delta E = \dfrac{h^2}{8mL^2}(n^2 - n_2)$. $L = \sqrt{\dfrac{h^2(n_1^2 - n_2)}{8m\Delta E}} = \sqrt{\dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})^2(2^2-1^2)}{8(9.11\times10^{-31}\ \text{kg})(1.63\times10^{-20}\ \text{J})}} = 3.33\times10^{-10}\ \text{m}$.

(b) The ground state energy for an electron in a box of the calculated dimensions is

$E = \dfrac{h^2}{8mL^2} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})^2}{8(9.11\times10^{-31}\ \text{kg})(3.33\times10^{-10}\ \text{m})^2} = 5.43\times10^{-19}\ \text{J} = 3.40\ \text{eV}$ (one-third of the original photon energy), which does not correspond to the $-13.6\ \text{eV}$ ground state energy of the hydrogen atom. Note that the energy levels for a particle in a box are proportional to n^2, whereas the energy levels for the hydrogen atom are proportional to $-\dfrac{1}{n^2}$.

40.5. **IDENTIFY** and **SET UP:** Eq.(40.9) gives the energy levels. Use this to obtain an expression for $E_2 - E_1$ and use the value given for this energy difference to solve for *L*.

EXECUTE: Ground state energy is $E_1 = \frac{h^2}{8mL^2}$; first excited state energy is $E_2 = \frac{4h^2}{8mL^2}$. The energy separation between these two levels is $\Delta E = E_2 - E_1 = \frac{3h^2}{8mL^2}$. This gives $L = h\sqrt{\frac{3}{8m\Delta E}} =$

$$L = 6.626\times10^{-34}\ \text{J}\cdot\text{s}\sqrt{\frac{3}{8(9.109\times10^{-31}\ \text{kg})(3.0\ \text{eV})(1.602\times10^{-19}\ \text{J/1 eV})}} = 6.1\times10^{-10}\ \text{m} = 0.61\ \text{nm}.$$

EVALUATE: This energy difference is typical for an atom and *L* is comparable to the size of an atom.

40.6. **(a)** The wave function for $n = 1$ vanishes only at $x = 0$ and $x = L$ in the range $0 \le x \le L$.

(b) In the range for x, the sine term is a maximum only at the middle of the box, $x = L/2$.

(c) The answers to parts (a) and (b) are consistent with the figure.

40.7. **IDENTIFY** and **SET UP:** For the $n = 2$ first excited state the normalized wave function is given by Eq.(40.13). $\psi_2(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$. $|\psi_2(x)|^2\,dx = \frac{2}{L}\sin^2\left(\frac{2\pi x}{L}\right)dx$. Examine $|\psi_2(x)|^2\,dx$ and find where it is zero and where it is maximum.

EXECUTE: **(a)** $|\psi_2|^2\,dx = 0$ implies $\sin\left(\frac{2\pi x}{L}\right) = 0$

$\frac{2\pi x}{L} = m\pi,\ \ m = 0,\ 1,\ 2, \ldots;\ x = m(L/2)$

For $m = 0,\ x = 0$; for $m = 1,\ x = L/2$; for $m = 2,\ x = L$

The probability of finding the particle is zero at $x = 0,\ L/2$, and L.

(b) $|\psi_2|^2\,dx$ is maximum when $\sin\left(\frac{2\pi x}{L}\right) = \pm1$

$\frac{2\pi x}{L} = m(\pi/2),\ m = 1,\ 3,\ 5, \ldots;\ x = m(L/4)$

For $m = 1,\ x = L/4$; for $m = 3,\ x = 3L/4$

The probability of finding the particle is largest at $x = L/4$ and $3L/4$.

(c) EVALUATE: The answers to part (a) correspond to the zeros of $|\psi|^2$ shown in Fig.40.5 in the textbook and the answers to part (b) correspond to the two values of *x* where $|\psi|^2$ in the figure is maximum.

40.8. $\frac{d^2\psi}{dx^2} = -k^2\psi$, and for ψ to be a solution of Eq.(40.3), $k^2 = E\frac{8\pi^2 m}{h^2} = E\frac{2m}{\hbar^2}$.

(b) The wave function must vanish at the rigid walls; the given function will vanish at $x = 0$ for any k, but to vanish at $x = L,\ kL = n\pi$ for integer n.

40.9. **(a) IDENTIFY** and **SET UP:** $\psi = A\cos kx$. Calculate $d\psi^2/dx^2$ and substitute into Eq.(40.3) to see if this equation is satisfied.

EXECUTE: Eq.(40.3): $-\frac{h^2}{8\pi^2 m}\frac{d^2\psi}{dx^2} = E\psi$

$\frac{d\psi}{dx} = A(-k\sin kx) = -Ak\sin kx$

$\frac{d^2\psi}{dx^2} = -Ak(k\cos kx) = -Ak^2\cos kx$

Thus Eq.(40.3) requires $-\frac{h^2}{8\pi^2 m}(-Ak^2\cos kx) = E(A\cos kx)$.

This says $-\frac{h^2k^2}{8\pi^2 m} = E$; $k = \frac{\sqrt{2mE}}{(h/2\pi)} = \frac{\sqrt{2mE}}{\hbar}$

$\psi = A\cos kx$ is a solution to Eq.(40.3) if $k = \frac{\sqrt{2mE}}{\hbar}$.

(b) EVALUATE: The wave function for a particle in a box with rigid walls at $x = 0$ and $x = L$ must satisfy the boundary conditions $\psi = 0$ at $x = 0$ and $\psi = 0$ at $x = L$. $\psi(0) = A\cos 0 = A$, since $\cos 0 = 1$. Thus ψ is not 0 at $x = 0$ and this wave function isn't acceptable because it doesn't satisfy the required boundary condition, even though it is a solution to the Schrödinger equation.

40.10. **(a)** The third excited state is $n = 4$, so

$$\Delta E = (4^2 - 1)\frac{h^2}{8mL^2} = \frac{15(6.626\times10^{-34}\ \text{J}\cdot\text{s})^2}{8(9.11\times10^{-31}\ \text{kg})(0.125\times10^{-9}\ \text{m})^2} = 5.78\times10^{-17}\ \text{J} = 361\ \text{eV}.$$

(b) $\lambda = \dfrac{hc}{\Delta E} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(3.0\times10^8\ \text{m/s})}{5.78\times10^{-17}\ \text{J}} = 3.44\ \text{nm}$

40.11. Recall $\lambda = \dfrac{h}{p} = \dfrac{h}{\sqrt{2mE}}$.

(a) $E_1 = \dfrac{h^2}{8mL^2} \Rightarrow \lambda_1 = \dfrac{h}{\sqrt{2mh^2/8mL^2}} = 2L = 2(3.0\times10^{-10}\ \text{m}) = 6.0\times10^{-10}\ \text{m}$. The wavelength is twice the width of the box. $p_1 = \dfrac{h}{\lambda_1} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})}{6.0\times10^{-10}\ \text{m}} = 1.1\times10^{-24}\ \text{kg}\cdot\text{m/s}$

(b) $E_2 = \dfrac{4h^2}{8mL^2} \Rightarrow \lambda_2 = L = 3.0\times10^{-10}\ \text{m}$. The wavelength is the same as the width of the box.

$p_2 = \dfrac{h}{\lambda_2} = 2p_1 = 2.2\times10^{-24}\ \text{kg}\cdot\text{m/s}$.

(c) $E_3 = \dfrac{9h^2}{8mL^2} \Rightarrow \lambda_3 = \dfrac{2}{3}L = 2.0\times10^{-10}\ \text{m}$. The wavelength is two-thirds the width of the box.

$p_3 = 3p_1 = 3.3\times10^{-24}\ \text{kg}\cdot\text{m/s}$.

40.12. **IDENTIFY:** If the given wave function is a solution to the Schrödinger equation, we will get an identity when we substitute that wave function into the Schrödinger equation.

SET UP: We must substitute the equation $\Psi(x,t) = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{n\pi x}{L}\right)e^{-iE_nt/\hbar}$ into the one-dimensional Schrödinger equation $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$.

EXECUTE: Taking the second derivative of $\Psi(x,t)$ with respect to x gives $\dfrac{d^2\Psi(x,\ t)}{dx^2} = -\left(\dfrac{n\pi}{L}\right)^2\Psi(x,t)$

Substituting this result into $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$, we get $\dfrac{\hbar^2}{2m}\left(\dfrac{n\pi}{L}\right)^2\Psi(x,t) = E\Psi(x,t)$ which gives $E_n = \dfrac{\hbar^2}{2m}\left(\dfrac{n\pi}{L}\right)^2$, the energies of a particle in a box.

EVALUATE: Since this process gives us the energies of a particle in a box, the given wave function is a solution to the Schrödinger equation.

40.13. **(a)** Eq.(40.1): $\dfrac{-\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + U\psi = E\psi$.

Left-hand side: $\dfrac{-\hbar^2}{2m}\dfrac{d^2}{dx^2}(A\sin kx) + U_0A\sin kx = \dfrac{\hbar^2k^2}{2m}A\sin kx + U_0A\sin kx = \left(\dfrac{\hbar^2k^2}{2m} + U_0\right)\psi$.

But $\dfrac{\hbar^2k^2}{2m} + U_0 > U_0 > E$ for constant k. But $\dfrac{\hbar^2k^2}{2m} + U_0$ should equal $E \Rightarrow$ no solution.

(b) If $E > U_0$, then $\dfrac{\hbar^2k^2}{2m} + U_0 = E$ is consistent and so $\psi = A\sin kx$ is a solution of Eq.(40.1) for this case.

40.14. According to Eq.(40.17), the wavelength of the electron inside of the square well is given by $k = \dfrac{\sqrt{2mE}}{\hbar} \Rightarrow \lambda_{\text{in}} = \dfrac{h}{\sqrt{2m(3U_0)}}$. By an analysis similar to that used to derive Eq.40.17, we can show that outside the box

$$\lambda_{\text{out}} = \frac{h}{\sqrt{2m(E-U_0)}} = \frac{h}{\sqrt{2m(2U_0)}}.$$

Thus, the ratio of the wavelengths is $\dfrac{\lambda_{\text{out}}}{\lambda_{\text{in}}} = \dfrac{\sqrt{2m(3U_0)}}{\sqrt{2m(2U_0)}} = \sqrt{\dfrac{3}{2}}$.

40.15. $E_1 = 0.625E_\infty = 0.625\dfrac{\pi^2\hbar^2}{2mL^2}$; $E_1 = 2.00\text{ eV} = 3.20\times10^{-19}\text{ J}$

$$L = \pi\hbar\left(\frac{0.625}{2(9.109\times10^{-31}\text{ kg})(3.20\times10^{-19}\text{ J})}\right)^{1/2} = 3.43\times10^{-10}\text{ m}$$

40.16. Since $U_0 = 6E_\infty$ we can use the result $E_1 = 0.625E_\infty$ from Section 40.3, so $U_0 - E_1 = 5.375E_\infty$ and the maximum wavelength of the photon would be

$$\lambda = \frac{hc}{U_0 - E_1} = \frac{hc}{(5.375)(h^2/8mL^2)} = \frac{8mL^2c}{(5.375)h}$$

$$\lambda = \frac{8(9.11\times10^{-31}\text{ kg})(1.50\times10^{-9}\text{ m})^2(3.00\times10^8\text{ m/s})}{(5.375)(6.63\times10^{-34}\text{ J}\cdot\text{s})} = 1.38\times10^{-6}\text{ m.}$$

40.17. Eq.(40.16): $\psi = A\sin\dfrac{\sqrt{2mE}}{\hbar}x + B\cos\dfrac{\sqrt{2mE}}{\hbar}x$

$$\frac{d^2\psi}{dx^2} = -A\left(\frac{2mE}{\hbar^2}\right)\sin\frac{\sqrt{2mE}}{\hbar}x - B\left(\frac{2mE}{\hbar^2}\right)\cos\frac{\sqrt{2mE}}{\hbar}x = \frac{-2mE}{\hbar^2}(\psi) = \text{Eq.(40.15).}$$

40.18. $\dfrac{d\psi}{dx} = \kappa(Ce^{\kappa x} - De^{-\kappa x})$, $\dfrac{d^2\psi}{dx^2} = \kappa^2(Ce^{\kappa x} + De^{-\kappa x}) = \kappa^2\psi$ for all constants C and D. Hence ψ is a solution to Eq.(40.1) for $-\dfrac{\hbar^2}{2m}\kappa^2 + U_0 = E$, or $\kappa = [2m(U_0 - E)]^{1/2}/\hbar$, and κ is real for $E < U_0$.

40.19. **IDENTIFY:** Find the transition energy ΔE and set it equal to the energy of the absorbed photon. Use $E = hc/\lambda$ to find the wavelength of the photon.

SET UP: $U_0 = 6E_\infty$, as in Fig.40.8 in the textbook, so $E_1 = 0.625E_\infty$ and $E_3 = 5.09E_\infty$ with $E_\infty = \dfrac{\pi^2\hbar^2}{2mL^2}$. In this problem the particle bound in the well is a proton, so $m = 1.673\times10^{-27}$ kg.

EXECUTE: $E_\infty = \dfrac{\pi^2\hbar^2}{2mL^2} = \dfrac{\pi^2(1.055\times10^{-34}\text{ J}\cdot\text{s})^2}{2(1.673\times10^{-27}\text{ kg})(4.0\times10^{-15}\text{ m})^2} = 2.052\times10^{-12}\text{ J}$. The transition energy is

$\Delta E = E_3 - E_1 = (5.09 - 0.625)E_\infty = 4.465E_\infty$. $\Delta E = 4.465(2.052\times10^{-12}\text{ J}) = 9.162\times10^{-12}\text{ J}$

The wavelength of the photon that is absorbed is related to the transition energy by $\Delta E = hc/\lambda$, so

$$\lambda = \frac{hc}{\Delta E} = \frac{(6.626\times10^{-34}\text{ J}\cdot\text{s})(2.998\times10^8\text{ m/s})}{9.162\times10^{-12}\text{ J}} = 2.2\times10^{-14}\text{ m} = 22\text{ fm.}$$

EVALUATE: The wavelength of the photon is comparable to the size of the box.

40.20. **IDENTIFY:** The longest wavelength corresponds to the smallest energy change.

SET UP: The ground level energy level of the infinite well is $E_\infty = \dfrac{h^2}{8mL^2}$, and the energy of the photon must be equal to the energy difference between the two shells.

EXECUTE: The 400.0 nm photon must correspond to the $n = 1$ to $n = 2$ transition. Since $U_0 = 6E_\infty$, we have $E_2 = 2.43E_\infty$ and $E_1 = 0.625E_\infty$. The energy of the photon is equal to the energy difference between the two levels, and $E_\infty = \dfrac{h^2}{8mL^2}$, which gives $E_\gamma = E_2 - E_1 \Rightarrow \dfrac{hc}{\lambda} = (2.43 - 0.625)E_\infty = \dfrac{1.805\ h^2}{8mL^2}$

Solving for L gives $L = \sqrt{\dfrac{(1.805)h\lambda}{8mc}} = \sqrt{\dfrac{(1.805)(6.626\times10^{-34}\text{ J}\cdot\text{s})(4.00\times10^{-7}\text{ m})}{8(9.11\times10^{-31}\text{ kg})(3.00\times10^8\text{ m/s})}} = 4.68\times10^{-10}\text{ m} = 0.468\text{ nm.}$

EVALUATE: This width is approximately half that of a Bohr hydrogen atom.

40.21. $T = 16\dfrac{E}{U_0}\left(1 - \dfrac{E}{U_0}\right)e^{-2L\sqrt{2m(U_0-E)}/\hbar}$. $\dfrac{E}{U_0} = \dfrac{6.0\text{ eV}}{11.0\text{ eV}}$ and $E - U_0 = 5\text{ eV} = 8.0\times10^{-19}\text{ J}$.

(a) $L = 0.80\times10^{-9}$ m: $T = 16\left(\dfrac{6.0\text{ eV}}{11.0\text{ eV}}\right)\left(1 - \dfrac{6.0\text{ ev}}{11.0\text{ eV}}\right)e^{-2(0.80\times10^{-9}\text{ m})\sqrt{2(9.11\times10^{-31}\text{ kg})(8.0\times10^{-19}\text{ J})}/1.055\times10^{-34}\text{ J}\cdot\text{s}} = 4.4\times10^{-8}$

(b) $L = 0.40\times10^{-9}$ m: $T = 4.2\times10^{-4}$.

40.22. The transmission coefficient is $T = 16\dfrac{E}{U_0}\left(1-\dfrac{E}{U_0}\right)e^{-2\sqrt{2m(U_0-E)}L/\hbar}$, with $E = 5.0\text{ eV}, L = 0.60\times10^{-9}\text{ m},$ and $m = 9.11\times10^{-31}\text{ kg}$

(a) $U_0 = 7.0\text{ eV} \Rightarrow T = 5.5\times10^{-4}$.

(b) $U_0 = 9.0\text{ eV} \Rightarrow T = 1.8\times10^{-5}$

(c) $U_0 = 13.0\text{ eV} \Rightarrow T = 1.1\times10^{-7}$.

40.23. **IDENTIFY** and **SET UP:** Use Eq.(39.1), where $K = p^2/2m$ and $E = K + U$.

EXECUTE: $\lambda = h/p = h/\sqrt{2mK}$, so $\lambda\sqrt{K}$ is constant

$\lambda_1\sqrt{K_1} = \lambda_2\sqrt{K_2}$; λ_1 and K_1 are for $x > L$ where $K_1 = 2U_0$ and λ_2 and K_2 are for $0 < x < L$ where $K_2 = E - U_0 = U_0$

$$\frac{\lambda_1}{\lambda_2} = \sqrt{\frac{K_2}{K_1}} = \sqrt{\frac{U_0}{2U_0}} = \frac{1}{\sqrt{2}}$$

EVALUATE: When the particle is passing over the barrier its kinetic energy is less and its wavelength is larger.

40.24. **IDENTIFY:** The probability of tunneling depends on the energy of the particle and the width of the barrier.

SET UP: The probability of tunneling is approximately $T = Ge^{-2\kappa L}$, where $G = 16\dfrac{E}{U_0}\left(1-\dfrac{E}{U_0}\right)$ and $\kappa = \dfrac{\sqrt{2m(U_0-E)}}{\hbar}$.

EXECUTE: $G = 16\dfrac{E}{U_0}\left(1-\dfrac{E}{U_0}\right) = 16\dfrac{50.0\text{ eV}}{70.0\text{ eV}}\left(1-\dfrac{50.0\text{ eV}}{70.0\text{ eV}}\right) = 3.27.$

$$\kappa = \frac{\sqrt{2m(U_0-E)}}{\hbar} = \frac{\sqrt{2(1.67\times10^{-27}\text{ kg})(70.0\text{ eV}-50.0\text{ eV})(1.60\times10^{-19}\text{ J/eV})}}{(6.63\times10^{-34}\text{ J}\cdot\text{s})/2\pi} = 9.8\times10^{11}\text{ m}^{-1}$$

Solving $T = Ge^{-2\kappa L}$ for L gives $L = \dfrac{1}{2\kappa}\ln(G/T) = \dfrac{1}{2(9.8\times10^{11}\text{ m}^{-1})}\ln\left(\dfrac{3.27}{0.0030}\right) = 3.6\times10^{-12}\text{ m} = 3.6\text{ pm}$

If the proton were replaced with an electron, the electron's mass is much smaller so L would be larger.

EVALUATE: An electron can tunnel through a much wider barrier than a proton of the same energy.

40.25. **IDENTIFY** and **SET UP:** The probability is $T = Ae^{-2\kappa L}$, with $A = 16\dfrac{E}{U_0}\left(1-\dfrac{E}{U_0}\right)$ and $\kappa = \dfrac{\sqrt{2m(U_0-E)}}{\hbar}$.

$E = 32\text{ eV}, U_0 = 41\text{ eV}, L = 0.25\times10^{-9}\text{ m}.$ Calculate T.

EXECUTE: **(a)** $A = 16\dfrac{E}{U_0}\left(1-\dfrac{E}{U_0}\right) = 16\dfrac{32}{41}\left(1-\dfrac{32}{41}\right) = 2.741.$

$$\kappa = \frac{\sqrt{2m(U_0-E)}}{\hbar}$$

$$\kappa = \frac{\sqrt{2(9.109\times10^{-31}\text{ kg})(41\text{ eV}-32\text{ eV})(1.602\times10^{-19}\text{ J/eV})}}{1.055\times10^{-34}\text{ J}\cdot\text{s}} = 1.536\times10^{10}\text{ m}^{-1}$$

$T = Ae^{-2\kappa L} = (2.741)e^{-2(1.536\times10^{10}\text{ m}^{-1})(0.25\times10^{-9}\text{ m})} = 2.741e^{-7.68} = 0.0013$

(b) The only change in the mass m, which appears in κ.

$$\kappa = \frac{\sqrt{2m(U_0-E)}}{\hbar}$$

$$\kappa = \frac{\sqrt{2(1.673\times10^{-27}\text{ kg})(41\text{ eV}-32\text{ eV})(1.602\times10^{-19}\text{ J/eV})}}{1.055\times10^{-34}\text{ J}\cdot\text{s}} = 6.584\times10^{11}\text{ m}^{-1}$$

Then $T = Ae^{-2\kappa L} = (2.741)e^{-2(6.584\times10^{11}\text{ m}^{-1})(0.25\times10^{-9}\text{ m})} = 2.741e^{-392.2} = 10^{-143}$

EVALUATE: The more massive proton has a much smaller probability of tunneling than the electron does.

40.26. $T = Ge^{-2\kappa L}$ with $G = 16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)$ and $\kappa = \frac{\sqrt{2m(U_0-E)}}{\hbar}$, so $T = 16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)e^{\frac{-2\sqrt{2m(U_0-E)}}{\hbar}L}$.

(a) If $U_0 = 30.0\times10^6$ eV, $L = 2.0\times10^{-15}$ m, $m = 6.64\times10^{-27}$ kg and $U_0 - E = 1.0\times10^6$ eV ($E = 29.0\times10^6$ eV), $T = 0.090$.

(b) If $U_0 - E = 10.0\times10^6$ eV ($E = 20.0\times10^6$ eV), $T = 0.014$.

40.27. **IDENTIFY** and **SET UP:** The energy levels are given by Eq.(40.26), where $\omega = \sqrt{\frac{k'}{m}}$.

EXECUTE: $\omega = \sqrt{\frac{k'}{m}} = \sqrt{\frac{110\text{ N/m}}{0.250\text{ kg}}} = 21.0$ rad/s

The ground state energy is given by Eq.(40.26):

$E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}(1.055\times10^{-34}\text{ J}\cdot\text{s})(21.0\text{ rad/s}) = 1.11\times10^{-33}\text{ J}(1\text{ eV}/1.602\times10^{-19}\text{ J}) = 6.93\times10^{-15}$ eV

$E_n = \left(n+\frac{1}{2}\right)\hbar\omega;\ E_{(n+1)} = \left(n+1+\frac{1}{2}\right)\hbar\omega$

The energy separation between these adjacent levels is

$\Delta E = E_{n+1} - E_n = \hbar\omega = 2E_0 = 2(1.11\times10^{-33}\text{ J}) = 2.22\times10^{-33}\text{ J} = 1.39\times10^{-14}$ eV

EVALUATE: These energies are extremely small; quantum effects are not important for this oscillator.

40.28. Let $\sqrt{mk'}/2\hbar = \delta$, and so $\frac{d\psi}{dx} = -2x\delta\psi$ and $\frac{d^2\psi}{dx^2} = (4x^2\delta^2 - 2\delta)\psi$, and ψ is a solution of Eq.(40.21) if

$E = \frac{\hbar^2}{m}\delta = \frac{1}{2}\hbar\sqrt{k'/m} = \frac{1}{2}\hbar\omega.$

40.29. **IDENTIFY:** We can model the molecule as a harmonic oscillator. The energy of the photon is equal to the energy difference between the two levels of the oscillator.

SET UP: The energy of a photon is $E_\gamma = hf = hc/\lambda$, and the energy levels of a harmonic oscillator are given by

$E_n = \left(n+\frac{1}{2}\right)\hbar\sqrt{\frac{k'}{m}} = \left(n+\frac{1}{2}\right)\hbar\omega.$

EXECUTE: **(a)** The photon's energy is $E_\gamma = \frac{hc}{\lambda} = \frac{(6.63\times10^{-34}\text{ J}\cdot\text{s})(3.00\times10^8\text{ m/s})}{5.8\times10^{-6}\text{ m}} = 0.21$ eV

(b) The transition energy is $\Delta E = E_{n+1} - E_n = \hbar\omega = \hbar\sqrt{\frac{k'}{m}}$, which gives $\frac{2\pi\hbar c}{\lambda} = \hbar\sqrt{\frac{k'}{m}}$. Solving for k', we get

$k' = \frac{4\pi^2c^2m}{\lambda^2} = \frac{4\pi^2(3.00\times10^8\text{ m/s})^2(5.6\times10^{-26}\text{ kg})}{(5.8\times10^{-6}\text{ m})^2} = 5{,}900$ N/m.

EVALUATE: This would be a rather strong spring in the physics lab.

40.30. According to Eq.(40.26), the energy released during the transition between two adjacent levels is twice the ground state energy $E_3 - E_2 = \hbar\omega = 2E_0 = 11.2$ eV.

For a photon of energy E

$$E = hf \Rightarrow \lambda = \frac{c}{f} = \frac{hc}{E} = \frac{(6.63\times10^{-34}\text{ J}\cdot\text{s})(3.00\times10^8\text{ m/s})}{(11.2\text{ eV})(1.60\times10^{-19}\text{ J/eV})} = 111\text{ nm}.$$

40.31. **IDENTIFY** and **SET UP:** Use the energies given in Eq.(40.26) to solve for the amplitude A and maximum speed v_{max} of the oscillator. Use these to estimate Δx and Δp_x and compute the uncertainty product $\Delta x\Delta p_x$.

EXECUTE: The total energy of a Newtonian oscillator is given by $E = \frac{1}{2}k'A^2$ where k' is the force constant and A is the amplitude of the oscillator. Set this equal to the energy $E = (n+\frac{1}{2})\hbar\omega$ of an excited level that has quantum number n, where $\omega = \sqrt{\frac{k'}{m}}$, and solve for A: $\frac{1}{2}k'A^2 = (n+\frac{1}{2})\hbar\omega$

$A = \sqrt{\frac{(2n+1)\hbar\omega}{k'}}$

The total energy of the Newtonian oscillator can also be written as $E = \frac{1}{2}mv_{max}^2$. Set this equal to $E = (n+\frac{1}{2})\hbar\omega$ and solve for v_{max}: $\frac{1}{2}mv_{max}^2 = (n+\frac{1}{2})\hbar\omega$

$v_{max} = \sqrt{\frac{(2n+1)\hbar\omega}{m}}$

Thus the maximum linear momentum of the oscillator is $p_{\max} = mv_{\max} = \sqrt{(2n+1)\hbar m\omega}$. Assume that A represents the uncertainty Δx in position and that $p_{\max}$ is the corresponding uncertainty Δp_x in momentum. Then the uncertainty product is $\Delta x \Delta p_x = \sqrt{\frac{(2n+1)\hbar\omega}{k'}}\sqrt{(2n+1)\hbar m\omega} = (2n+1)\hbar\omega\sqrt{\frac{m}{k'}} = (2n+1)\hbar\omega\left(\frac{1}{\omega}\right) = (2n+1)\hbar.$

EVALUATE: For $n = 1$ this gives $\Delta x \Delta p_x = 3\hbar$, in agreement with the result derived in Section 40.4. The uncertainty product $\Delta x \Delta p_x$ increases with n.

40.32. **(a)** $\frac{|\psi(A)|^2}{|\psi(0)|^2} = \exp\left(-\frac{\sqrt{mk'}}{\hbar}A^2\right) = \exp\left(-\sqrt{mk'}\frac{\omega}{k'}\right) = e^{-1} = 0.368.$

This is consistent with what is shown in Figure 40.20 in the textbook.

(b) $\frac{|\psi(2A)^2|}{|\psi(0)|^2} = \exp\left(-\frac{\sqrt{mk'}}{\hbar}(2A)^2\right) = \exp\left(-\sqrt{mk'}4\frac{\omega}{k'}\right) = e^{-4} = 1.83\times10^{-2}.$

This figure cannot be read this precisely, but the qualitative decrease in amplitude with distance is clear.

40.33. **IDENTIFY:** We model the atomic vibration in the crystal as a harmonic oscillator.

SET UP: The energy levels of a harmonic oscillator are given by $E_n = \left(n+\frac{1}{2}\right)\hbar\sqrt{\frac{k'}{m}} = \left(n+\frac{1}{2}\right)\hbar\omega.$

EXECUTE: **(a)** The ground state energy of a simple harmonic oscillator is

$$E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar\sqrt{\frac{k'}{m}} = \frac{(1.055\times10^{-34}\text{ J}\cdot\text{s})}{2}\sqrt{\frac{12.2\text{ N/m}}{3.82\times10^{-26}\text{ kg}}} = 9.43\times10^{-22}\text{ J} = 5.89\times10^{-3}\text{ eV}$$

(b) $E_4 - E_3 = \hbar\omega = 2E_0 = 0.0118\text{ eV}$, so $\lambda = \frac{hc}{E} = \frac{(6.63\times10^{-34}\text{ J}\cdot\text{s})(3.00\times10^8\text{ m/s})}{1.88\times10^{-21}\text{ J}} = 106\ \mu\text{m}$

(c) $E_{n+1} - E_n = \hbar\omega = 2E_0 = 0.0118\text{ eV}$

EVALUATE: These energy differences are much smaller than those due to electron transitions in the hydrogen atom.

40.34. **IDENTIFY:** If the given wave function is a solution to the Schrödinger equation, we will get an identity when we substitute that wave function into the Schrödinger equation.

SET UP: The given function is $\psi(x) = Ae^{ikx}$, and the one-dimensional Schrödinger equation is

$-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x).$

EXECUTE: Start with the given function and take the indicated derivatives: $\psi(x) = Ae^{ikx}$. $\frac{d\psi(x)}{dx} = Aike^{ikx}$.

$\frac{d^2\psi(x)}{dx^2} = Ai^2k^2e^{ikx} = -Ak^2e^{ikx}$. $\frac{d^2\psi(x)}{dx^2} = -k^2\psi(x)$. $-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2} = \frac{\hbar^2}{2m}\ k^2\psi(x)$. Substituting these results into the one-dimensional Schrödinger equation gives $\frac{\hbar^2k^2}{2m}\ \psi(x) + U_0\psi(x) = E\ \psi(x)$.

EVALUATE: $\psi(x) = A\ e^{ikx}$ is a solution to the one-dimensional Schrödinger equation if $E - U_0 = \frac{\hbar^2k^2}{2m}$ or $k = \sqrt{\frac{2m(E-U_0)}{\hbar^2}}$. (Since $U_0 < E$ was given, k is the square root of a positive quantity.) In terms of the particle's momentum p: $k = p/\hbar$, and in terms of the particle's de Broglie wavelength λ: $k = 2\pi/\lambda$.

40.35. **IDENTIFY:** Let I refer to the region $x < 0$ and let II refer to the region $x > 0$, so $\psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x}$ and $\psi_{II}(x) = Ce^{ik_2x}$. Set $\psi_I(0) = \psi_{II}(0)$ and $\frac{d\psi_I}{dx} = \frac{d\psi_{II}}{dx}$ at $x = 0$.

SET UP: $\frac{d}{dx}(e^{ikx}) = ike^{ikx}$.

EXECUTE: $\psi_I(0) = \psi_{II}(0)$ gives $A + B = C$. $\frac{d\psi_I}{dx} = \frac{d\psi_{II}}{dx}$ at $x = 0$ gives $ik_1A - ik_1B = ik_2C$. Solving this pair of equations for B and C gives $B = \left(\frac{k_1-k_2}{k_1+k_2}\right)A$ and $C = \left(\frac{2k_2}{k_1+k_2}\right)A$.

EVALUATE: The probability of reflection is $R = \dfrac{B^2}{A^2} = \dfrac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$. The probability of transmission is $T = \dfrac{C^2}{A^2} = \dfrac{4k_1^2}{(k_1 + k_2)^2}$. Note that $R + T = 1$.

40.36. **(a)** $R_n = \dfrac{(n+1)^2 - n^2}{n^2} = \dfrac{2n+1}{n^2} = \dfrac{2}{n} + \dfrac{1}{n^2}$. This is never larger than it is for $n = 1$, and $R_1 = 3$.

(b) R approaches zero; in the classical limit, there is no quantization, and the spacing of successive levels is vanishingly small compared to the energy levels.

40.37. **IDENTIFY** and **SET UP:** The energy levels are given by Eq.(40.9): $E_n = \dfrac{n^2 h^2}{8mL^2}$. Calculate ΔE for the transition and set $\Delta E = hc/\lambda$, the energy of the photon.

EXECUTE: **(a)** Ground level, $n = 1$, $E_1 = \dfrac{h^2}{8mL^2}$

First excited level, $n = 2$, $E_2 = \dfrac{4h^2}{8mL^2}$

The transition energy is $\Delta E = E_2 - E_1 = \dfrac{3h^2}{8mL^2}$. Set the transition energy equal to the energy hc/λ of the emitted photon. This gives $\dfrac{hc}{\lambda} = \dfrac{3h^2}{8mL^2}$.

$$\lambda = \frac{8mcL^2}{3h} = \frac{8(9.109\times10^{-31}\ \text{kg})(2.998\times10^{8}\ \text{m/s})(4.18\times10^{-9}\ \text{m})^2}{3(6.626\times10^{-34}\ \text{J}\cdot\text{s})}$$

$\lambda = 1.92\times10^{-5}\ \text{m} = 19.2\ \mu\text{m}.$

(b) Second excited level has $n = 3$ and $E_3 = \dfrac{9h^2}{8mL^2}$. The transition energy is $\Delta E = E_3 - E_2 = \dfrac{9h^2}{8mL^2} - \dfrac{4h^2}{8mL^2} = \dfrac{5h^2}{8mL^2}$.

$\dfrac{hc}{\lambda} = \dfrac{5h^2}{8mL^2}$ so $\lambda = \dfrac{8mcL^2}{5h} = \dfrac{3}{5}(19.2\ \mu\text{m}) = 11.5\ \mu\text{m}.$

EVALUATE: The energy spacing between adjacent levels increases with *n*, and this corresponds to a shorter wavelength and more energetic photon in part (b) than in part (a).

40.38. **(a)** $\dfrac{2}{L}\displaystyle\int_0^{L/4} \sin^2\frac{\pi x}{L}dx = \frac{2}{L}\int_0^{L/4}\frac{1}{2}\left(1-\cos\frac{2\pi x}{L}\right)dx = \frac{1}{L}\left(x - \frac{L}{2\pi}\sin\frac{2\pi x}{L}\right)_0^{L/4} = \frac{1}{4} - \frac{1}{2\pi}$, which is about 0.0908.

(b) Repeating with limits of $L/4$ and $L/2$ gives $\dfrac{1}{L}\left(x - \dfrac{L}{2\pi}\sin\dfrac{2\pi x}{L}\right)_{L/4}^{L/2} = \dfrac{1}{4} + \dfrac{1}{2\pi}$, about 0.409.

(c) The particle is much likely to be nearer the middle of the box than the edge.

(d) The results sum to exactly 1/2, which means that the particle is as likely to be between $x = 0$ and $L/2$ as it is to be between $x = L/2$ and $x = L$.

(e) These results are represented in Figure 40.5b in the textbook.

40.39. **IDENTIFY:** The probability of the particle being between x_1 and x_2 is $\int_{x_1}^{x_2} |\psi|^2 dx$, where ψ is the normalized wave function for the particle.

(a) SET UP: The normalized wave function for the ground state is $\psi_1 = \sqrt{\dfrac{2}{L}}\sin\left(\dfrac{\pi x}{L}\right)$.

EXECUTE: The probability *P* of the particle being between $x = L/4$ and $x = 3L/4$ is $P = \displaystyle\int_{L/4}^{3L/4}|\psi_1|^2 dx = \frac{2}{L}\int_{L/4}^{3L/4}\sin^2\left(\frac{\pi x}{L}\right)dx$. Let $y = \pi x/L$; $dx = (L/\pi)dy$ and the integration limits become $\pi/4$ and $3\pi/4$.

$$P = \frac{2}{L}\left(\frac{L}{\pi}\right)\int_{\pi/4}^{3\pi/4}\sin^2 y\,dy = \frac{2}{\pi}\left[\frac{1}{2}y - \frac{1}{4}\sin 2y\right]_{\pi/4}^{3\pi/4}$$

$$P = \frac{2}{\pi}\left[\frac{3\pi}{8} - \frac{\pi}{8} - \frac{1}{4}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{4}\sin\left(\frac{\pi}{2}\right)\right]$$

$P = \dfrac{2}{\pi}\left(\dfrac{\pi}{4} - \dfrac{1}{4}(-1) + \dfrac{1}{4}(1)\right) = \dfrac{1}{2} + \dfrac{1}{\pi} = 0.818.$ (Note: The integral formula $\int\sin^2 y\,dy = \dfrac{1}{2}y - \dfrac{1}{4}\sin 2y$ was used.)

(b) SET UP: The normalized wave function for the first excited state is $\psi_2 = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$

EXECUTE: $P = \int_{L/4}^{3L/4} |\psi_2|^2 dx = \frac{2}{L}\int_{L/4}^{3L/4} \sin^2\left(\frac{2\pi x}{L}\right) dx.$ Let $y = 2\pi x/L$; $dx = (L/2\pi)dy$ and the integration limits become $\pi/2$ and $3\pi/2$.

$$P = \frac{2}{L}\left(\frac{L}{2\pi}\right)\int_{\pi/2}^{3\pi/2} \sin^2 y\, dy = \frac{1}{\pi}\left[\frac{1}{2}y - \frac{1}{4}\sin 2y\right]_{\pi/2}^{3\pi/2} = \frac{1}{\pi}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = 0.500$$

(c) EVALUATE: These results are consistent with Fig.40.4b in the textbook. That figure shows that $|\psi|^2$ is more concentrated near the center of the box for the ground state than for the first excited state; this is consistent with the answer to part (a) being larger than the answer to part (b). Also, this figure shows that for the first excited state half the area under $|\psi|^2$ curve lies between $L/4$ and $3L/4$, consistent with our answer to part (b).

40.40. Using the normalized wave function $\psi_1 = \sqrt{2/L}\,\sin(\pi x/L)$, the probabilities $|\psi|^2\, dx$ are

(a) $(2/L)\sin^2(\pi/4)dx = dx/L$

(b) $(2/L)\sin^2(\pi/2)dx = 2dx/L$

(c) $(2/L)\sin^2(3\pi/4) = dx/L.$

40.41. **IDENTIFY** and **SET UP:** The normalized wave function for the $n = 2$ first excited level is $\psi_2 = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right).$

$P = |\psi(x)|^2\, dx$ is the probability that the particle will be found in the interval x to $x + dx$.

EXECUTE: **(a)** $x = L/4$

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\left(\frac{2\pi}{L}\right)\left(\frac{L}{4}\right)\right) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{2}\right) = \sqrt{\frac{2}{L}}.$$

$P = (2/L)dx$

(b) $x = L/2$

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\left(\frac{2\pi}{L}\right)\left(\frac{L}{2}\right)\right) = \sqrt{\frac{2}{L}}\sin(\pi) = 0$$

$P = 0$

(c) $x = 3L/4$

$$\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\left(\frac{2\pi}{L}\right)\left(\frac{3L}{4}\right)\right) = \sqrt{\frac{2}{L}}\sin\left(\frac{3\pi}{2}\right) = -\sqrt{\frac{2}{L}}.$$

$P = (2/L)dx$

EVALUATE: Our results are consistent with the $n = 2$ part of Fig.40.5 in the textbook. $|\psi|^2$ is zero at the center of the box and is symmetric about this point.

40.42. $\Delta\vec{p} = \vec{p}_{\text{final}} - \vec{p}_{\text{initial}}$. $|\vec{p}| = \hbar k = \frac{\hbar n\pi}{L} = \frac{hn}{2L}$. At $x = 0$ the initial momentum at the wall is $\vec{p}_{\text{initial}} = -\frac{hn}{2L}\hat{i}$ and the final momentum, after turning around, is $\vec{p}_{\text{final}} = +\frac{hn}{2L}\hat{i}$. So, $\Delta\vec{p} = +\frac{hn}{2L}\hat{i} - \left(-\frac{hn}{2L}\hat{i}\right) = +\frac{hn}{L}\hat{i}$. At $x = L$ the initial momentum is $\vec{p}_{\text{initial}} = +\frac{hn}{2L}\hat{i}$ and the final momentum, after turning around, is $\vec{p}_{\text{final}} = -\frac{hn}{2L}\hat{i}$. So,

$$\Delta\vec{p} = -\frac{hn}{2L}\hat{i} - \frac{hn}{2L}\hat{i} = -\frac{hn}{L}\hat{i}$$

40.43. **(a)** For a free particle, $U(x) = 0$ so Schrödinger's equation becomes $\frac{d^2\psi(x)}{dx^2} = -\frac{2m}{\hbar^2}E\psi(x)$. The graph is given in Figure 40.43.

(b) For $x < 0$: $\psi(x) = e^{+\kappa x}$. $\frac{d\psi(x)}{dx} = \kappa e^{+\kappa x}$. $\frac{d^2\psi(x)}{dx} = \kappa^2 e^{+\kappa x}$. So $\kappa^2 = -\frac{2m}{\hbar^2}E \Rightarrow E = -\frac{\hbar^2\kappa^2}{2m}$.

(c) For $x > 0$: $\psi(x) = e^{-\kappa x}$. $\frac{d\psi(x)}{dx} = -ke^{-\kappa x}$. $\frac{d^2\psi(x)}{dx} = \kappa^2 e^{-\kappa x}$

So again $\kappa^2 = -\frac{2m}{\hbar^2}E \Rightarrow E = \frac{-\hbar^2\kappa^2}{2m}$. Parts (c) and (d) show $\psi(x)$ satisfies the Schrödinger's equation, provided $E = \frac{-\hbar^2\kappa^2}{2m}$.

(d) Note $\frac{d\psi(x)}{dx}$ is discontinuous at $x = 0$. (That is, negative for $x > 0$ and positive for $x < 0$.)

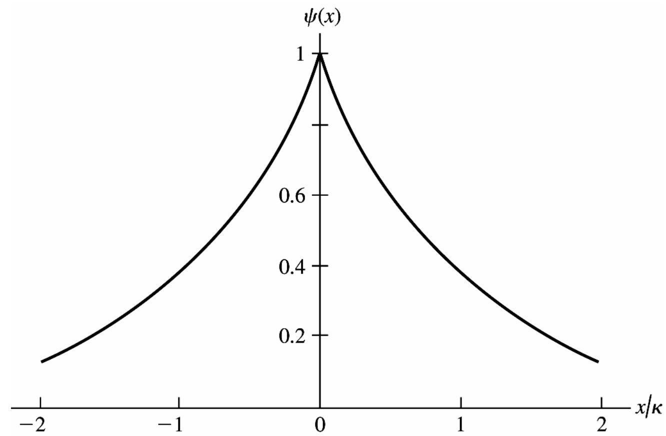

Figure 40.43

40.44. **IDENTIFY:** We start with the penetration distance formula given in the problem.

SET UP: The given formula is $\eta = \frac{\hbar}{\sqrt{2m(U_0 - E)}}$.

EXECUTE: **(a)** Substitute the given numbers into the formula:

$$\eta = \frac{\hbar}{\sqrt{2m(U_0 - E)}} = \frac{1.055\times10^{-34}\ \text{J}\cdot\text{s}}{\sqrt{2(9.11\times10^{-31}\ \text{kg})(20\ \text{eV} - 13\ \text{eV})(1.602\times10^{-19}\ \text{J/eV})}} = 7.4\times10^{-11}\ \text{m}$$

(b) $$\eta = \frac{1.055\times10^{-34}\ \text{J}\cdot\text{s}}{\sqrt{2(1.67\times10^{-27}\ \text{kg})(30\ \text{MeV} - 20\ \text{MeV})(1.602\times10^{-13}\ \text{J/MeV})}} = 1.44\times10^{-15}\ \text{m}$$

EVALUATE: The penetration depth varies widely depending on the mass and energy of the particle.

40.45. **(a)** We set the solutions for inside and outside the well equal to each other at the well boundaries, $x = 0$ and L.

$x = 0: A\sin(0) + B = C \Rightarrow B = C$, since we must have $D = 0$ for $x < 0$.

$x = L$: $A\sin\frac{\sqrt{2mE}L}{\hbar} + B\cos\frac{\sqrt{2mE}L}{\hbar} = +De^{-\kappa L}$ since $C = 0$ for $x > L$.

This gives $A\sin kL + B\cos kL = De^{-\kappa L}$, where $k = \frac{\sqrt{2mE}}{\hbar}$.

(b) Requiring continuous derivatives at the boundaries yields

$x = 0$: $\frac{d\psi}{dx} = kA\cos(k\cdot 0) - kB\sin(k\cdot 0) = kA = \kappa Ce^{k\cdot 0} \Rightarrow kA = \kappa C$

$x = L$: $kA\cos kL - kB\sin kL = -\kappa De^{-\kappa L}$.

40.46. $T = Ge^{-2\kappa L}$ with $G = 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right)$ and $\kappa = \frac{\sqrt{2m(U_0 - E)}}{\hbar} \Rightarrow L = -\frac{1}{2\kappa}\ln\left(\frac{T}{G}\right)$.

If $E = 5.5$ eV, $U_0 = 10.0$ eV, $m = 9.11\times10^{-31}$ kg, and $T = 0.0010$. Then

$$\kappa = \frac{\sqrt{2(9.11\times10^{-31}\ \text{kg})(4.5\ \text{eV})(1.60\times10^{-19}\ \text{J/eV})}}{(1.054\times10^{-34}\ \text{J}\cdot\text{s})} = 1.09\times10^{10}\ \text{m}^{-1} \text{ and } G = 16\frac{5.5\ \text{eV}}{10.0\ \text{eV}}\left(1 - \frac{5.5\ \text{eV}}{10.0\ \text{eV}}\right) = 3.96$$

so $L = -\frac{1}{2(1.09\times10^{10}\ \text{m}^{-1})}\ln\left(\frac{0.0010}{3.96}\right) = 3.8\times10^{-10}\ \text{m} = 0.38\ \text{nm}$.

40.47. **IDENTIFY** and **SET UP:** When κL is large, then $e^{\kappa L}$ is large and $e^{-\kappa L}$ is small. When κL is small, $\sinh\kappa L \to \kappa L$. Consider both κL large and κL small limits.

EXECUTE: **(a)** $T = \left[1 + \frac{(U_0 \sinh\kappa L)^2}{4E(U_0 - E)}\right]^{-1}$

$\sinh \kappa L = \dfrac{e^{\kappa L} - e^{-\kappa L}}{2}$

For $\kappa L \gg 1$, $\sinh \kappa L \to \dfrac{e^{\kappa L}}{2}$ and $T \to \left[1 + \dfrac{U_0^2 e^{2\kappa L}}{16E(U_0 - E)}\right]^{-1} = \dfrac{16E(U_0 - E)}{16E(U_0 - E) + U_0^2 e^{2\kappa L}}$

For $\kappa L \gg 1$, $16E(U_0 - E) + U_0^2 e^{2\kappa L} \to U_0^2 e^{2\kappa L}$

$T \to \dfrac{16E(U_0 - E)}{U_0^2 e^{2\kappa L}} = 16\left(\dfrac{E}{U_0}\right)\left(1 - \dfrac{E}{U_0}\right)e^{-2\kappa L}$, which is Eq.(40.21).

(b) $\kappa L = \dfrac{L\sqrt{2m(U_0 - E)}}{\hbar}$. So $\kappa L \gg 1$ when L is large (barrier is wide) or $U_0 - E$ is large. (E is small compared to U_0.)

(c) $\kappa = \dfrac{\sqrt{2m(U_0 - E)}}{\hbar}$; κ becomes small as E approaches U_0. For κ small, $\sinh \kappa L \to \kappa L$ and

$T \to \left[1 + \dfrac{U_0^2 \kappa^2 L^2}{4E(U_0 - E)}\right]^{-1} = \left[1 + \dfrac{U_0^2 2m(U_0 - E)L^2}{\hbar^2 4E(U_0 - E)}\right]^{-1}$ (using the definition of κ)

Thus $T \to \left[1 + \dfrac{2U_0^2 L^2 m}{4E\hbar^2}\right]^{-1}$

$U_0 \to E$ so $\dfrac{U_0^2}{E} \to E$ and $T \to \left[1 + \dfrac{2EL^2 m}{4\hbar^2}\right]^{-1}$

But $k^2 = \dfrac{2mE}{\hbar^2}$, so $T \to \left[1 + \left(\dfrac{kL}{2}\right)^2\right]^{-1}$, as was to be shown.

EVALUATE: When κL is large Eq.(40.20) applies and T is small. When $E \to U_0$, T does not approach unity.

40.48. **(a)** $E = \dfrac{1}{2}mv^2 = (n + (1/2))\hbar\omega = (n + (1/2))hf$, and solving for n,

$$n = \frac{\frac{1}{2}mv^2}{hf} - \frac{1}{2} = \frac{(1/2)(0.020\ \text{kg})(0.360\ \text{m/s})^2}{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(1.50\ \text{Hz})} - \frac{1}{2} = 1.3\times10^{30}.$$

(b) The difference between energies is $\hbar\omega = hf = (6.63\times10^{-34}\ \text{J}\cdot\text{s})(1.50\ \text{Hz}) = 9.95\times10^{-34}$ J. This energy is too small to be detected with current technology

40.49. **IDENTIFY** and **SET UP:** Calculate the angular frequency ω of the pendulum and apply Eq.(40.26) for the energy levels.

EXECUTE: $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{0.500\ \text{s}} = 4\pi\ \text{s}^{-1}$

The ground-state energy is $E_0 = \dfrac{1}{2}\hbar\omega = \dfrac{1}{2}(1.055\times10^{-34}\ \text{J}\cdot\text{s})(4\pi\ \text{s}^{-1}) = 6.63\times10^{-34}$ J.

$E_0 = 6.63\times10^{-34}\ \text{J}(1\ \text{eV}/1.602\times10^{-19}\ \text{J}) = 4.14\times10^{-15}$ eV

$E_n = \left(n + \dfrac{1}{2}\right)\hbar\omega$

$E_{n+1} = \left(n + 1 + \dfrac{1}{2}\right)\hbar\omega$

The energy difference between the adjacent energy levels is
$\Delta E = E_{n+1} - E_n = \hbar\omega = 2E_0 = 1.33\times10^{-33}\ \text{J} = 8.30\times10^{-15}$ eV

EVALUATE: These energies are much too small to detect. Quantum effects are not important for ordinary size objects.

40.50. **IDENTIFY:** We model the electrons in the lattice as a particle in a box. The energy of the photon is equal to the energy difference between the two energy states in the box.

SET UP: The energy of an electron in the n^{th} level is $E_n = \dfrac{n^2h^2}{8mL^2}$. We do not know the initial or final levels, but we do know they differ by 1. The energy of the photon, hc/λ, is equal to the energy difference between the two states.

EXECUTE: The energy difference between the levels is $\Delta E = \dfrac{hc}{\lambda} = \dfrac{(6.63\times10^{-34}\ \text{J}\cdot\text{s})(3.00\times10^8\ \text{m/s})}{1.649\times10^{-7}\ \text{m}} =$
1.206×10^{-18} J. Using the formula for the energy levels in a box, this energy difference is equal to

$\Delta E = \left[n^2 - (n-1)^2\right]\dfrac{h^2}{8mL^2} = (2n-1)\dfrac{h^2}{8mL^2}$.

Solving for n gives $n=\left(\frac{\Delta E 8mL^2}{h^2}+1\right)=\frac{1}{2}\left(\frac{(1.206\times10^{-18}\text{ J})8(9.11\times10^{-31}\text{ kg})(0.500\times10^{-9}\text{ m})^2}{(6.626\times10^{-34}\text{ J}\cdot\text{s})^2}+1\right)=3.$

The transition is from $n=3$ to $n=2$.

EVALUATE: We know the transition is not from the $n=4$ to the $n=3$ state because we let n be the higher state and $n-1$ the lower state.

40.51. **IDENTIFY:** If the given wave function is a solution to the Schrödinger equation, we will get an identity when we substitute that wave function into the Schrödinger equation.

SET UP: The given wave function is $\psi_0(x)=A_0e^{-\alpha^2x^2/2}$ and the Schrödinger equation is

$$-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2}+\frac{k'x^2}{2}\psi(x)=E\ \psi(x).$$

EXECUTE: **(a)** Start by taking the derivatives: $\psi_0(x)=A_0e^{-\alpha^2x^2/2}$. $\frac{d\psi_0(x)}{dx}=-\alpha^2xA_0e^{-\alpha^2x^2/2}$.

$\frac{d^2\psi_0(x)}{dx^2}=-A_0\alpha^2e^{-\alpha^2x^2/2}+(\alpha^2)^2x^2A_0e^{-\alpha^2x^2/2}$. $\frac{d^2\psi_0(x)}{dx^2}=[-\alpha^2+(\alpha^2)^2x^2]\ \psi_0(x)$.

$-\frac{\hbar}{2m}\frac{d^2\psi_0(x)}{dx^2}=-\frac{\hbar^2}{2m}[-\alpha^2+(\alpha^2)^2x^2]\ \psi_0(x)$. Equation (40.22) is $-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2}+\frac{k'x^2}{2}\psi(x)=E\ \psi(x)$. Substituting the above result into that equation gives $-\frac{\hbar^2}{2m}[-\alpha^2+(\alpha^2)^2x^2]\ \psi_0(x)+\frac{k'x^2}{2}\psi_0(x)=E\ \psi_0(x)$. Since $\alpha^2=\frac{m\omega}{\hbar}$ and $\omega=\sqrt{\frac{k'}{m}}$, the coefficient of x^2 is $-\frac{\hbar^2}{2m}(\alpha^2)^2+\frac{k'}{2}=-\frac{\hbar^2}{2m}\left(\frac{m\omega}{\hbar}\right)^2+\frac{m\omega^2}{2}=0.$

(b) $A_0=\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}$

(c) The classical turning points are at $A=\pm\sqrt{\frac{\hbar}{\omega m}}$. The probability density function $|\psi|^2$ is

$|\psi_0(x)|^2=A_0^2e^{-\alpha^2x^2}=\left(\frac{m\omega}{\hbar\pi}\right)^{1/2}e^{-m\omega x^2/\hbar}$. At $x=0$, $|\psi_0|^2=\left(\frac{m\omega}{\hbar\pi}\right)^{1/2}$.

$\frac{d|\psi_0(x)|^2}{dx}=\left(\frac{m\omega}{\hbar\pi}\right)^{1/2}(-\alpha^2 2x)e^{-\alpha^2x^2}=-2\frac{m\omega}{\hbar}\left(\frac{m\omega}{\hbar\pi}\right)^{1/2}xe^{-\alpha^2x^2}$. At $x=0$, $\frac{d|\psi_0(x)|^2}{dx}=0.$

$\frac{d^2|\psi_0(x)|^2}{dx^2}=-2\frac{m\omega}{\hbar}\left(\frac{m\omega}{\hbar\pi}\right)^{1/2}[1-2\alpha^2x^2]e^{-\alpha^2x^2}$. At $x=0$, $\frac{d^2|\psi_0(x)|^2}{dx^2}<0$. Therefore, at $x=0$, the first derivative is zero and the second derivative is negative. Therefore, the probability density function has a maximum at $x=0$.

EVALUATE: $\psi_0(x)=A_0e^{-\alpha^2x^2/2}$ is a solution to equation (40.22) if $-\frac{\hbar^2}{2m}(-\alpha^2)\psi_0(x)=E\ \psi_0(x)$ or $E=\frac{\hbar^2\alpha^2}{2m}=\frac{\hbar\omega}{2}$. $E_0=\frac{\hbar\omega}{2}$ corresponds to $n=0$ in Equation (40.26).

40.52. **IDENTIFY:** If the given wave function is a solution to the Schrödinger equation, we will get an identity when we substitute that wave function into the Schrödinger equation.

SET UP: The given wave function is $\psi_1(x)=A_1 2xe^{-\alpha^2x^2/2}$ and the Schrödinger equation is

$$-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2}+\frac{k'x^2}{2}\psi(x)=E\ \psi(x).$$

EXECUTE: **(a)** Start by taking the indicated derivatives: $\psi_1(x)=A_1 2xe^{-\alpha^2x^2/2}$.

$\frac{d\psi_1(x)}{dx}=-2\alpha^2x^2A_1e^{-\alpha^2x^2/2}+2A_1e^{-\alpha^2x^2/2}$. $\frac{d^2\psi_1(x)}{dx^2}=-2A_1\alpha^2 2xe^{-\alpha^2x^2/2}-2A_1\alpha^2x^2(-\alpha^2x)e^{-\alpha^2x^2/2}+2A_1(-\alpha^2x)e^{-\alpha^2x^2/2}$.

$$\frac{d^2\psi_1(x)}{dx^2}=[-2\alpha^2+(\alpha^2)^2x^2-\alpha^2]\ \psi_1(x)=[-3\alpha^2+(\alpha^2)^2x^2]\ \psi_1(x)$$

$$-\frac{\hbar}{2m}\frac{d^2\psi_1(x)}{dx^2}=-\frac{\hbar^2}{2m}[-3\alpha^2+(\alpha^2)^2x^2]\ \psi_1(x)$$

Equation (40.22) is $-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2}+\frac{k'x^2}{2}\psi(x)=E\ \psi(x)$. Substituting the above result into that equation gives $-\frac{\hbar^2}{2m}[-3\alpha^2+(\alpha^2)^2x^2]\ \psi_1(x)+\frac{k'x^2}{2}\psi_1(x)=E\ \psi_1(x)$. Since $\alpha^2=\frac{m\omega}{\hbar}$ and $\omega=\sqrt{\frac{k'}{m}}$, the coefficient of x^2 is $-\frac{\hbar^2}{2m}(\alpha^2)^2+\frac{k'}{2}=-\frac{\hbar^2}{2m}\left(\frac{m\omega}{\hbar}\right)^2+\frac{m\omega^2}{2}=0$

(b) $A_1=\frac{1}{\sqrt{2}}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}$

(c) The probability density function $|\psi|^2$ is $|\psi_1(x)|^2=A_1^2 4x^2e^{-\alpha^2x^2}=\frac{1}{2}\left(\frac{m\omega}{\hbar\pi}\right)^{1/2}4x^2e^{-\frac{m\omega x^2}{\hbar}}$

At $x=0$, $|\psi_1|^2=0$. $\frac{d|\psi_1(x)|^2}{dx}=A_1^2 8xe^{-\alpha^2x^2}+A_1^2 4x^2(-\alpha^2 2x)e^{-\alpha^2x^2}=A_1^2 8xe^{-\alpha^2x^2}-A_1^2 8x^3\alpha^2e^{-\alpha^2x^2}$

At $x=0$, $\frac{d|\psi_1(x)|^2}{dx}=0$. At $x=\pm\frac{1}{\alpha}$, $\frac{d|\psi_1(x)|^2}{dx}=0$.

$\frac{d^2|\psi_1(x)|^2}{dx^2}=A_1^2 8e^{-\alpha^2x^2}+A_1^2 8x(-\alpha^2 2x)e^{-\alpha^2x^2}-A_1^2 8(3x^2)\alpha^2e^{-\alpha^2x^2}-A_1^2 8x^3\alpha^2(-\alpha^2 2x)e^{-\alpha^2x^2}$.

$\frac{d^2|\psi_1(x)|^2}{dx^2}=A_1^2 8e^{-\alpha^2x^2}-A_1^2 16x^2\alpha^2e^{-\alpha^2x^2}-A_1^2 24x^2\alpha^2e^{-\alpha^2x^2}+A_1^2 16x^4(\alpha^2)^2e^{-\alpha^2x^2}$. At $x=0$, $\frac{d^2|\psi_1(x)|^2}{dx^2}>0$. So at $x=0$, the first derivative is zero and the second derivative is positive. Therefore, the probability density function has a minimum at $x=0$. At $x=\pm\frac{1}{\alpha}$, $\frac{d^2|\psi_1(x)|^2}{dx^2}<0$. So at $x=\pm\frac{1}{\alpha}$, the first derivative is zero and the second derivative is negative. Therefore, the probability density function has maxima at $x=\pm\frac{1}{\alpha}$, corresponding to the classical turning points for $n=0$ as found in the previous question.

EVALUATE: $\psi_1(x)=A_1 2xe^{-\alpha^2x^2/2}$ is a solution to equation (40.22) if $-\frac{\hbar^2}{2m}(-3\alpha^2)\psi_1(x)=E\ \psi_1(x)$ or $E=\frac{3\hbar^2\alpha^2}{2m}=\frac{3\hbar\omega}{2}$. $E_1=\frac{3\hbar\omega}{2}$ corresponds to $n=1$ in Equation (40.26).

40.53. IDENTIFY and **SET UP:** Evaluate $\partial^2\psi/\partial x^2$, $\partial^2\psi/\partial y^2$, and $\partial^2\psi/\partial z^2$ for the proposed ψ and put Eq.(40.29). Use that ψ_{n_x}, ψ_{n_y}, and ψ_{n_z} are each solutions to Eq.(40.22).

EXECUTE: **(a)** $-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+U\psi=E\psi$

ψ_{n_x}, ψ_{n_y}, ψ_{n_z} are each solutions of Eq.(40.22), so $-\frac{\hbar^2}{2m}\frac{d^2\psi_{n_x}}{dx^2}+\frac{1}{2}k'x^2\psi_{n_x}=E_{n_x}\psi_{n_x}$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{n_y}}{dy^2}+\frac{1}{2}k'y^2\psi_{n_y}=E_{n_y}\psi_{n_y}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{n_z}}{dz^2}+\frac{1}{2}k'z^2\psi_{n_z}=E_{n_z}\psi_{n_z}$$

$\psi=\psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z)$, $U=\frac{1}{2}k'x^2+\frac{1}{2}k'y^2+\frac{1}{2}k'z^2$

$$\frac{\partial^2\psi}{\partial x^2}=\left(\frac{d^2\psi_{n_x}}{dx^2}\right)\psi_{n_y}\psi_{n_z},\ \frac{\partial^2\psi}{\partial y^2}=\left(\frac{d^2\psi_{n_y}}{dy^2}\right)\psi_{n_x}\psi_{n_z},\ \frac{\partial^2\psi}{\partial z^2}=\left(\frac{d^2\psi_{n_z}}{dz^2}\right)\psi_{n_x}\psi_{n_y}.$$

So $-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+U\psi=\left(-\frac{\hbar^2}{2m}\frac{d^2\psi_{n_x}}{dx^2}+\frac{1}{2}k'x^2\psi_{n_x}\right)\psi_{n_y}\psi_{n_z}$

$$+\left(-\frac{\hbar^2}{2m}\frac{d^2\psi_{n_y}}{dy^2}+\frac{1}{2}k'y^2\psi_{n_y}\right)\psi_{n_x}\psi_{n_z}+\left(-\frac{\hbar^2}{2m}\frac{d^2\psi_{n_z}}{dz^2}+\frac{1}{2}k'z^2\psi_{n_z}\right)\psi_{n_x}\psi_{n_y}$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+U\psi=(E_{n_x}+E_{n_y}+E_{n_z})\psi$$

Therefore, we have shown that this ψ is a solution to Eq.(40.29), with energy

$$E_{n_x n_y n_z} = E_{n_x} + E_{n_y} + E_{n_z} = \left(n_x + n_y + n_z + \frac{3}{2}\right)\hbar\omega$$

(b) and **(c)** The ground state has $n_x = n_y = n_z = 0,$ so the energy is $E_{000} = \frac{3}{2}\hbar\omega.$ There is only one set of n_x, n_y and n_z that give this energy.

First-excited state: $n_x = 1,\ n_y = n_z = 0$ or $n_y = 1,\ n_x = n_z = 0$ or $n_z = 1,\ n_x = n_y = 0$ and $E_{100} = E_{010} = E_{001} = \frac{5}{2}\hbar\omega$

There are three different sets of $n_x,\ n_y,\ n_z$ quantum numbers that give this energy, so there are three different quantum states that have this same energy.

EVALUATE: For the three-dimensional isotropic harmonic oscillator, the wave function is a product of one-dimensional harmonic oscillator wavefunctions for each dimension. The energy is a sum of energies for three one-dimensional oscillators. All the excited states are degenerate, with more than one state having the same energy.

40.54. $\omega_1 = \sqrt{k_1'/m},\ \omega_2 = \sqrt{k_2'/m}.$ Let $\psi_{n_x}(x)$ be a solution of Eq.(40.22) with $E_{n_x} = \left(n_x + \frac{1}{2}\right)\hbar\omega_1, \psi_{n_x}(y)$ be a similar solution, $\psi_{n_z}(z)$ be a solution of Eq.(40.22) but with z as the independent variable instead of x, and energy $E_{n_z} = \left(n_z + \frac{1}{2}\right)\omega_2.$

(a) As in Problem 40.53, look for a solution of the form $\psi(x, y, z) = \psi_{n_x}(x)\psi_{n_y}(y)\psi_{n_z}(z)$. Then,

$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = \left(E_{n_x} - \frac{1}{2}k_1'x^2\right)\psi$ with similar relations for $\frac{\partial^2\psi}{\partial y^2}$ and $\frac{\partial^2\psi}{\partial z^2}$. Adding,

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) = \left(E_{n_x} + E_{n_y} + E_{n_z} - \frac{1}{2}k_1'x^2 - \frac{1}{2}k_1'y^2 - \frac{1}{2}k_2'z^2\right)\psi$$

$$= (E_{n_x} + E_{n_y} + E_{n_z} - U)\psi = (E - U)\psi$$

where the energy E is $E = E_{n_x} + E_{n_y} + E_{n_z} = \hbar\left[(n_x + n_y + 1)\omega_1^2 + \left(n_z + \frac{1}{2}\right)\omega_2^2\right]$,with n_x, n_y and n_z all nonnegative integers.

(b) The ground level corresponds to $n_x = n_y = n_z = 0,$ and $E = \hbar\left(\omega_1^2 + \frac{1}{2}\omega_2^2\right)$. The first excited level corresponds to $n_x = n_y = 0$and $n_z = 1,$ since $\omega_1^2 > \omega_2^2,$ and $E = \hbar\omega\left(\omega_1^2 + \frac{3}{2}\omega_2^2\right)$. There is only one set of quantum numbers for both the ground state and the first excited state.

40.55. **(a)** $\psi(x) = A \sin kx$ and $\psi(-L/2) = 0 = \psi(+L/2)$

$$\Rightarrow 0 = A \sin\left(\frac{+kL}{2}\right) \Rightarrow \frac{+kL}{2} = n\pi \Rightarrow k = \frac{2n\pi}{L} = \frac{2\pi}{\lambda}$$

$$\Rightarrow \lambda = \frac{L}{n} \Rightarrow p_n = \frac{h}{\lambda n} = \frac{nh}{L} \Rightarrow E_n = \frac{p_n^2}{2m} = \frac{n^2h^2}{2mL^2} = \frac{(2n)^2h^2}{8mL^2},\ \text{where } n = 1, 2...$$

(b) $\psi(x) = A \cos kx$ and $\psi(-L/2) = 0 = \psi(+L/2)$

$$\Rightarrow 0 = A \cos\left(\frac{kL}{2}\right) \Rightarrow \frac{kL}{2} = (2n+1)\frac{\pi}{2} \Rightarrow k = \frac{(2n+1)\pi}{L} = \frac{2\pi}{\lambda}$$

$$\Rightarrow \lambda = \frac{2L}{(2n+1)} \Rightarrow p_n = \frac{(2n+1)h}{2L}$$

$$\Rightarrow E_n = \frac{(2n+1)^2h^2}{8mL^2}\ n = 0, 1, 2...$$

(c) The combination of all the energies in parts (a) and (b) is the same energy levels as given in Eq.(40.9), where $E_n = \frac{n^2h^2}{8mL^2}$.

(d) Part (a)'s wave functions are odd, and part (b)'s are even.

40.56. **(a)** As with the particle in a box, $\psi(x) = A \sin kx$, where A is a constant and $k^2 = 2mE/\hbar^2$. Unlike the particle in a box, however, k and hence E do not have simple forms.

(b) For $x > L$, the wave function must have the form of Eq.(40.18). For the wave function to remain finite as $x \to \infty$, $C = 0$. The constant $\kappa^2 = 2m(U_0 - E)/\hbar$, as in Eq.(14.17) and Eq.(40.18).

(c) At $x = L$, $A \sin kL = De^{-\kappa L}$ and $kA \cos kL = -\kappa De^{-\kappa L}$. Dividing the second of these by the first gives $k \cot kL = -\kappa$, a transcendental equation that must be solved numerically for different values of the length L and the ratio E/U_0.

40.57. **(a)** $E = K + U(x) = \dfrac{p^2}{2m} + U(x) \Rightarrow p = \sqrt{2m(E - U(x))}$. $\lambda = \dfrac{h}{p} \Rightarrow \lambda(x) = \dfrac{h}{\sqrt{2m(E - U(x))}}$.

(b) As $U(x)$ gets larger (i.e., $U(x)$ approaches E from below—recall $k \geq 0$), $E - U(x)$ gets smaller, so $\lambda(x)$ gets larger.

(c) When $E = U(x)$, $E - U(x) = 0$, so $\lambda(x) \to \infty$.

(d) $\displaystyle\int_a^b \frac{dx}{\lambda(x)} = \int_a^b \frac{dx}{h/\sqrt{2m(E - U(x))}} = \frac{1}{h}\int_a^b \sqrt{2m(E - U(x))}\, dx = \frac{n}{2} \Rightarrow \int_a^b \sqrt{2m(E - U(x))}\, dx = \frac{hn}{2}.$

(e) $U(x) = 0$ for $0 < x < L$ with classical turning points at $x = 0$ and $x = L$. So, $\int_a^b \sqrt{2m(E - U(x))}\, dx = \int_0^L \sqrt{2mE}dx = \sqrt{2mE}\int_0^L dx = \sqrt{2mE}L$. So, from part (d),

$\sqrt{2mE}L = \dfrac{hn}{2} \Rightarrow E = \dfrac{1}{2m}\left(\dfrac{hn}{2L}\right)^2 = \dfrac{h^2n^2}{8mL^2}$.

(f) Since $U(x) = 0$ in the region between the turning points at $x = 0$ and $x = L$, the results is the *same* as part (e). The height U_0 never enters the calculation. WKB is best used with *smoothly* varying potentials $U(x)$.

40.58. **(a)** At the turning points $E = \dfrac{1}{2}k'x_{\text{TP}}^2 \Rightarrow x_{\text{TP}} = \pm\sqrt{\dfrac{2E}{k'}}$.

(b) $\displaystyle\int_{-\sqrt{2E/k'}}^{+\sqrt{2E/k'}} \sqrt{2m\left(E - \frac{1}{2}k'x^2\right)}dx = \frac{nh}{2}$. To evaluate the integral, we want to get it into a form that matches the standard integral given. $\sqrt{2m\left(E - \dfrac{1}{2}k'x^2\right)} = \sqrt{2mE - mk'x^2} = \sqrt{mk'}\sqrt{\dfrac{2mE}{mk'} - x^2} = \sqrt{mk'}\sqrt{\dfrac{2E}{k'} - x^2}$.

Letting $A^2 = \dfrac{2E}{k'}$, $a = -\sqrt{\dfrac{2E}{k'}}$, and $b = +\sqrt{\dfrac{2E}{k'}}$

$$\Rightarrow \sqrt{mk'}\int_a^b \sqrt{A^2 - x^2}\, dx = 2\frac{\sqrt{mk'}}{2}\left[x\sqrt{A^2 - x^2} + A^2 \arcsin\left(\frac{x}{|A|}\right)\right]\Bigg|_0^b$$

$$= \sqrt{mk'}\left[\sqrt{\frac{2E}{k'}}\sqrt{\frac{2E}{k'} - \frac{2E}{k'}} + \frac{2E}{k'}\arcsin\left(\frac{\sqrt{2E/k'}}{\sqrt{2E/k'}}\right)\right] = \sqrt{mk'}\,\frac{2E}{k'}\arcsin(1) = 2E\sqrt{\frac{m}{k'}}\left(\frac{\pi}{2}\right).$$

Using WKB, this is equal to $\dfrac{hn}{2}$, so $E\sqrt{\dfrac{m}{k'}}\pi = \dfrac{hn}{2}$. Recall $\omega = \sqrt{\dfrac{k'}{m}}$, so $E = \dfrac{h}{2\pi}\omega n = \hbar\omega n$.

(c) We are missing the zero-point-energy offset of $\dfrac{\hbar\omega}{2}$ $\left(\text{recall } E = \hbar\omega\left(n + \dfrac{1}{2}\right)\right)$. However, our approximation isn't bad at all!

40.59. **(a)** At the turning points $E = A|x_{\text{TP}}| \Rightarrow x_{\text{TP}} = \pm\dfrac{E}{A}$.

(b) $\displaystyle\int_{-E/A}^{+E/A} \sqrt{2m(E - A|x|)}dx = 2\int_0^{E/A} \sqrt{2m(E - Ax)}\, dx$. Let $y = 2m(E - Ax) \Rightarrow$

$dy = -2mA\, dx$ when $x = \dfrac{E}{A}$, $y = 0$, and when $x = 0$, $y = 2mE$. So

$\displaystyle 2\int_0^{\frac{E}{A}} \sqrt{2m(E - Ax)}dx = -\frac{1}{mA}\int_{2mE}^0 y^{1/2}dy = -\frac{2}{3mA}y^{3/2}\Big|_{2mE}^0 = \frac{2}{3mA}(2mE)^{3/2}$. Using WKB, this is equal to $\dfrac{hn}{2}$. So,

$\dfrac{2}{3mA}(2mE)^{3/2} = \dfrac{hn}{2} \Rightarrow E = \dfrac{1}{2m}\left(\dfrac{3mAh}{4}\right)^{2/3} n^{2/3}$.

(c) The difference in energy decreases between successive levels. For example:

$$1^{2/3} - 0^{2/3} = 1,\ 2^{2/3} - 1^{2/3} = 0.59,\ 3^{3/2} - 2^{3/2} = 0.49, \ldots$$

- A sharp ∞ step gave ever-increasing level differences $(\sim n^2)$.
- A parabola $(\sim x^2)$ gave evenly spaced levels ($\sim n$).
- Now, a linear potential $(\sim x)$ gives ever-decreasing level differences ($\sim n^{2/3}$).

Roughly speaking, if the curvature of the potential (~ second derivative) is bigger than that of a parabola, then the level differences will increase. If the curvature is less than a parabola, the differences will decrease.