Essential University Physics

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9 Systems of Particles

PowerPoint® Lecture prepared by Richard Wolfson

In this lecture you'll learn

- How to find the center of mass of a system of particles
- The principle of conservation of momentum, and how it applies to systems of particles
- How to analyze collisions
  - Inelastic collisions
  - Elastic collisions

Center of Mass

- The **center of mass** of a composite object or system of particles is the point where, from the standpoint of Newton's second law, the mass acts as though it were concentrated.
- The position of the center of mass is a weighted average of the positions of the individual particles:
  - For a system of discrete particles,

$$\vec{r}_{cm} = \frac{\sum m_i \cdot \vec{r}_i}{M}$$

  - For a continuous distribution of matter,

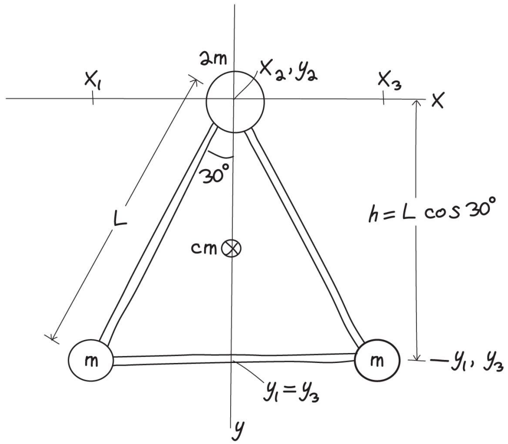
$$\vec{r}_{cm} = lim_{\Delta m_i \to 0} \frac{\sum \Delta m_i \cdot \vec{r}_i}{M} = \frac{\int \vec{r} \cdot dm}{M}$$

  - In both cases, *M* is the system's total mass.

Finding the center of mass

- A system of individual particles

$$\vec{r}_{cm} = \frac{\sum m_i \cdot \vec{r}_i}{M}$$

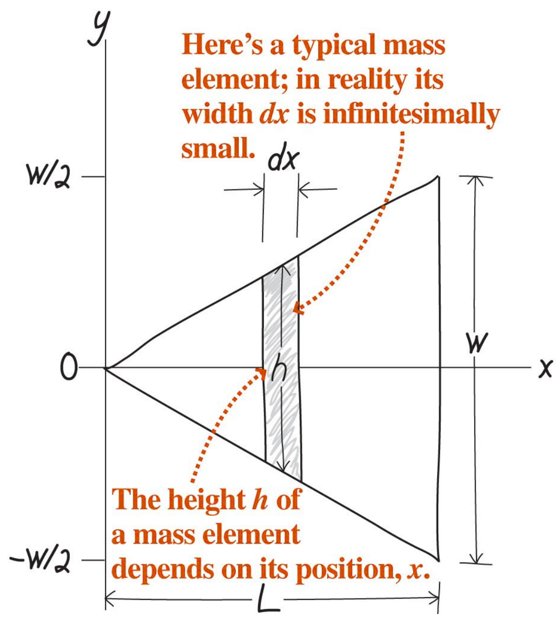


$$x_{\text{cm}} = \frac{mx_1 + mx_3}{4m} = \frac{m(x_1 - x_1)}{4m} = 0$$

$$y_{\text{cm}} = \frac{my_1 + my_3}{4m} = \frac{2my_1}{4m} = \frac{1}{2}y_1 = \frac{\sqrt{3}}{4}L = 0.43L$$

- A system of continuous matter

$$\vec{r}_{cm} = lim_{\Delta m_i \to 0} \frac{\sum \Delta m_i \cdot \vec{r}_i}{M} = \frac{\int \vec{r} \cdot dm}{M}$$



- Express the mass element dm in terms of the geometrical variable x:

$$\frac{dm}{M} = \frac{(w/L)x\ dx}{\frac{1}{2}wL} = \frac{2x\ dx}{L^2}$$

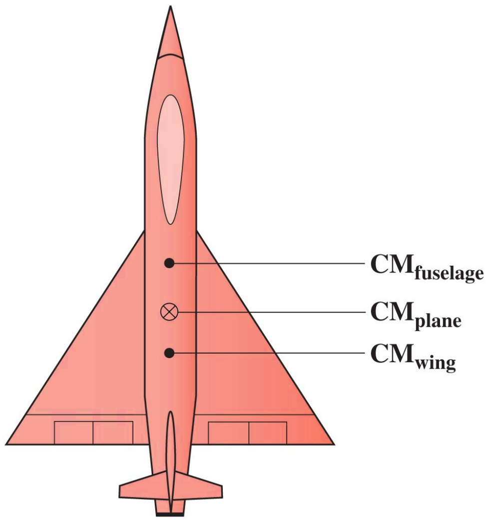
- Evaluate the integral:

$$x_{\text{cm}} = \frac{1}{M}\int x\ dm = \frac{1}{M}\int_0^L x\left(\frac{2Mx}{L^2}dx\right) = \frac{2}{L^2}\int_0^L x^2\ dx$$

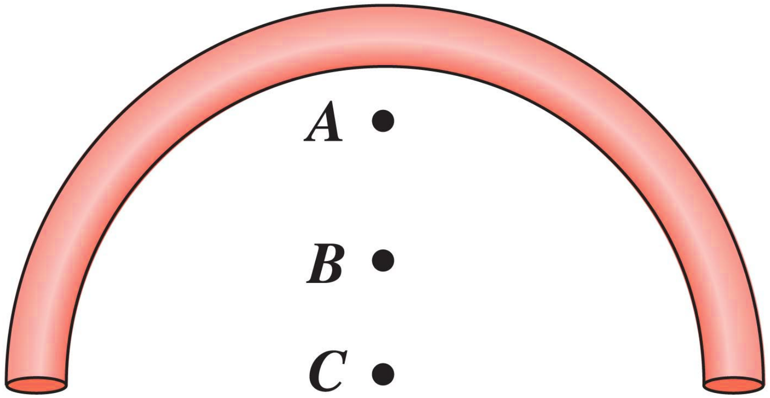
$$\text{so}\ \ x_{\text{cm}} = \frac{2}{L^2}\int_0^L x^2\ dx = \frac{2}{L^2}\frac{x^3}{3}\Bigg|_0^L = \frac{2L^3}{3L^2} = \frac{2}{3}L$$

More on center of mass

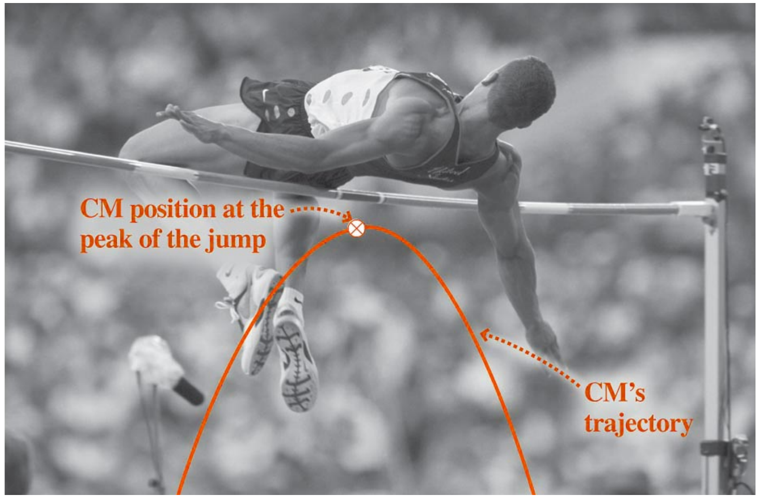
- The center of mass of a composite object can be found from the CMs of its individual parts.



- An object's center of mass need not lie within the object!
- Which point is the CM?



- The high jumper clears the bar, but his CM doesn't.



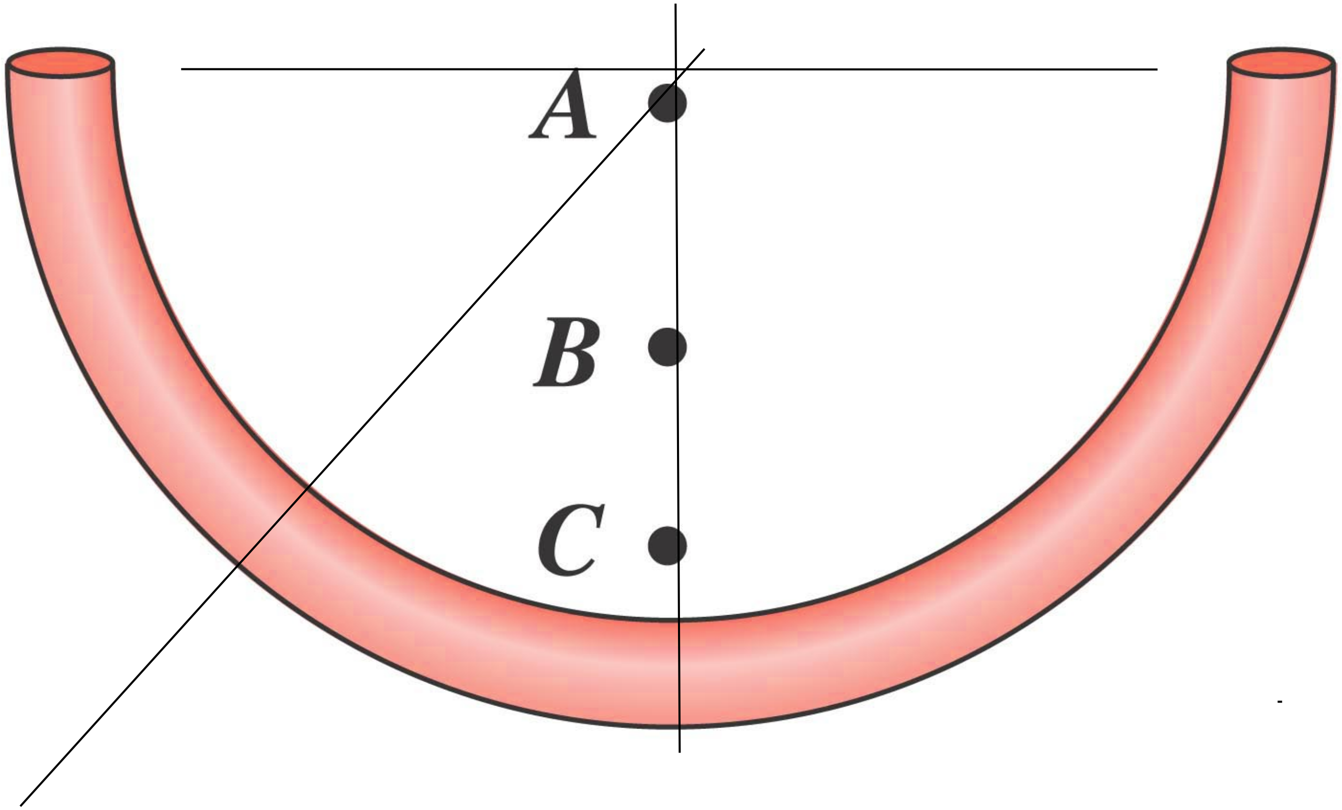
Clicker question

A thick wire is bent into a semicircle, as shown in the figure. Which of the points shown is the center of mass of the wire?

A. Point *A*

C. Point *B*

D. Point *C*



Motion of the center of mass

- The center of mass obeys Newton's second law:

$$\vec{F}_{\text{net external}} = M\vec{a}_{\text{cm}}$$

- Here most parts of the skier's body undergo complex motions, but his center of mass describes the parabolic trajectory of a projectile:

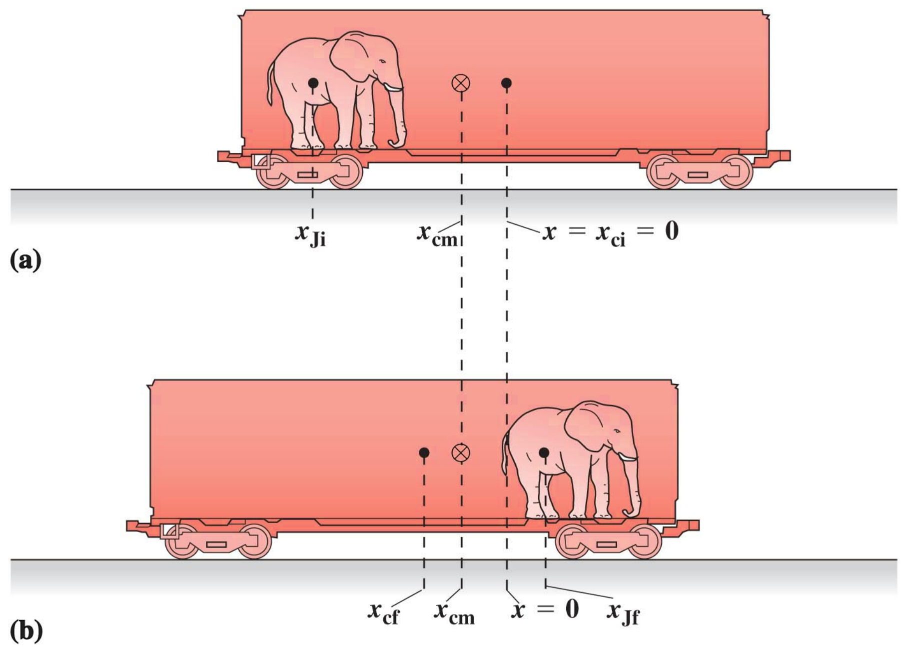
Motion of the center of mass

- Absent any *external* forces on a system, the center of mass motion remains unchanged; if it's at rest, it remains in the same place—no matter what *internal* forces may act.

Here Jumbo walks, but the CM of the rail car plus elephant doesn't move. This allows us to find the car's final position:

$$x_{cm} = \frac{m_J x_{Jf} + m_c x_{cf}}{M} = \frac{m_J(x_{Ji} + 19\text{ m} + x_{cf}) + m_c x_{cf}}{M}$$

$$x_{cm} = -\frac{(19\text{ m})m_J}{(m_J + m_c)} = -\frac{(19\text{ m})(4.8\text{ t})}{(15\text{ t} + 4.8\text{ t})} = -4.6\text{ m}$$



Momentum and the center of mass

- The center of mass obeys Newton's law, which can be written $\vec{F}_{net\ external} = M \cdot \vec{a}_{cm}$ or, equivalently,

$$\vec{F}_{net\ external} = \frac{d\vec{P}}{dt}$$

where $\vec{P}$ is the total momentum of the system:

$$\vec{P} = \sum m_i \cdot \vec{v}_i = M \cdot \vec{v}_{cm}$$

with $\vec{v}_{cm}$ the velocity of the center of mass.

Clicker question

A 500-g fireworks rocket is moving with velocity $\vec{v} = 60\hat{\boldsymbol{j}}$ m/s at the instant it explodes. If you were to add the momentum vectors of all its fragments just after the explosion, what would be the result?

A. $\vec{v} = 60\hat{\boldsymbol{j}}$ kg·m/s

C. $\vec{v} = 30\hat{\boldsymbol{j}}$ kg·m/s

D. $\vec{v} = 60000\hat{\boldsymbol{j}}$ kg·m/s

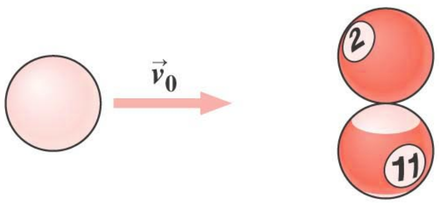
E. $\vec{v} = 30000\hat{\boldsymbol{j}}$ kg·m/s

Conservation of momentum

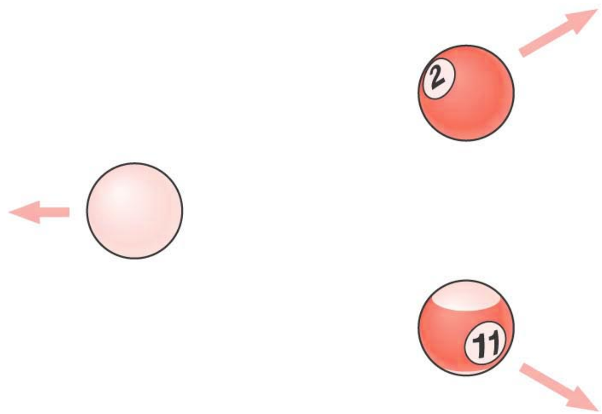
- When the net external force is zero, $\frac{d\vec{P}}{dt} = 0$.
- Therefore the total momentum of the system is unchanged: $\vec{P} = const.$

  This is the **conservation of linear momentum.**

- A system of three billiard balls:
  - Initially two are at rest; all the momentum is in the left-hand ball:



- Now they're all moving, but the total momentum remains the same:



Collisions

- A collision is a brief, intense interaction between objects.
  - The collision time is short compared with the timescale of the objects' overall motion.
  - Internal forces of the collision are so large that we can neglect any external forces acting on the system during the brief collision time.
  - Therefore linear momentum is essentially conserved during collisions.

Elastic and inelastic collisions

- In an **elastic collision**, the internal forces of the collision are conservative.
  - Therefore an elastic collision conserves kinetic energy as well as linear momentum.
- In an **inelastic collision**, the forces are not conservative and mechanical energy is lost.
  - In a **totally inelastic collision**, the colliding objects stick together to form a single composite object.
  - But if a collision is totally inelastic, that doesn't necessarily mean that all kinetic energy is lost.

Totally inelastic collisions

- Totally inelastic collisions are governed entirely by conservation of momentum.
  - Since the colliding objects join to form a single composite object, there's only one final velocity:

- Therefore conservation of momentum reads

$$m_1 \cdot \vec{v}_1 + m_2 \cdot \vec{v}_2 = (m_1 + m_2) \cdot \vec{v}_f$$

Elastic collisions

- Elastic collisions conserve both momentum and kinetic energy:

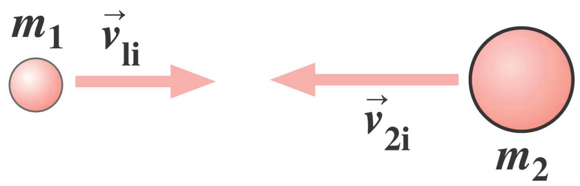
- Therefore the conservation laws read

$$m_1 \cdot \vec{v}_{1i} + m_2 \cdot \vec{v}_{2i} = m_1 \cdot \vec{v}_{1f} + m_2 \cdot \vec{v}_{2f}$$

$$\frac{1}{2} m_1 \cdot v^2_{1i} + \frac{1}{2} m_2 \cdot v^2_{2i} = \frac{1}{2} m_1 \cdot v^2_{1f} + \frac{1}{2} m_2 \cdot v^2_{2f}$$

Elastic collisions in one dimension

- In general, the conservation laws don't determine the outcome of an elastic collision.
  - Other information is needed, such as the direction of one of the outgoing particles.
- But for one-dimensional collisions, when particles collide head-on, then the initial velocities determine the outcome:



- Solving both conservation laws in this case gives

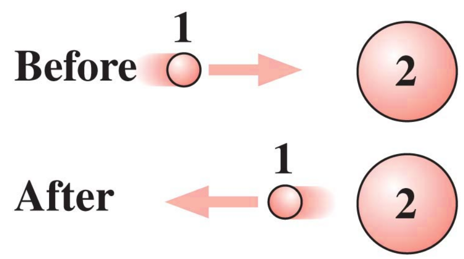
$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}$$

Special cases: 1-D elastic collisions; m_2 initially at rest

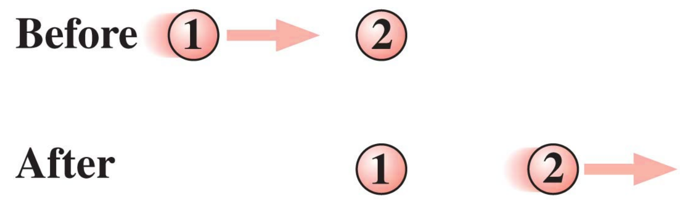
1) $m_1 << m_2$

Incident object rebounds with essentially its incident velocity



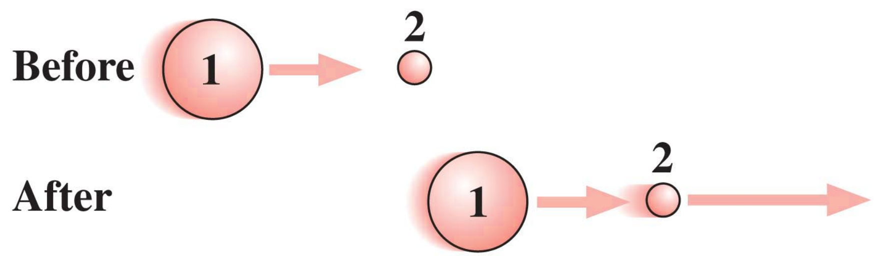
2) $m_1 = m_2$

Incident object stops; struck object moves away with initial speed of incident object



3) $m_1 >> m_2$

Incident object continues with essentially its initial velocity; struck object moves away with twice that velocity



Clicker question

Ball A is at rest on a level floor. Ball B collides elastically with Ball A, and the two move off separately, but in the same direction. What can you conclude about the masses of the two balls?

A. Ball A and Ball B have the same mass.

C. Ball B has a greater mass than Ball A.

D. Ball A has a greater mass than Ball B.

E. You cannot conclude anything without more information.

Summary

- A composite system behaves as though its mass is concentrated at the **center of mass:**

$$\vec{r}_{cm} = \frac{\sum m_i \cdot \vec{r}_i}{M} \text{ (discrete particles)} \qquad \vec{r}_{cm} = \frac{\int \vec{r} \cdot dm}{M} \text{ (continuous matter)}$$

- The center of mass obeys Newton's laws, so

$$\vec{F}_{net\ external} = M \cdot \vec{a}_{cm} \quad \text{or} \quad \vec{F}_{net\ external} = \frac{d\vec{P}}{dt}$$

- In the absence of a net external force, a system's linear momentum is conserved, regardless of what happens internally to the system.
- Collisions are brief, intense interactions that conserve momentum.
  - Elastic collisions also conserve kinetic energy.
  - Totally inelastic collisions occur when colliding objects join to make a single composite object.