Summary of Electricity and Magnetism

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A summary of Introduction to Electrodynamics by David J. Griffiths

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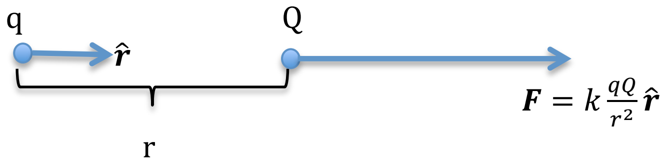
Electrostatics

Coulomb's Law

Coulomb's law gives the force [N] on a test charge Q due to a single point charge q, which is at rest at a distance r:

$$\boldsymbol{F} = k\frac{qQ}{r^2}\hat{\boldsymbol{r}}$$

Where $k = \frac{1}{4\pi\varepsilon_0}$ and $\hat{\boldsymbol{r}}$ equals a unit vector in the direction from q to Q.
The constant ε_0 is called the *permittivity of free space*.



If we have several point charges $q_1, q_2, \ldots, q_n$, at distances $r_1, r_2, \ldots, r_n$ respectively. The total force on Q is:

$$\boldsymbol{F} = \boldsymbol{F_1} + \boldsymbol{F_2} + \cdots + \boldsymbol{F_n} = kQ\sum_{i=1}^{n}\frac{q_i}{{r_i}^2}\hat{\boldsymbol{r}}_i$$

Electric Field

The electric field [N/C] is the force per unit charge that would be exerted on a test charge, if one were placed. So $\boldsymbol{F} = Q\boldsymbol{E}$, where

$$\boldsymbol{E}(\boldsymbol{r}) = k\sum_{i=1}^{n}\frac{q_i}{{r_i}^2}\hat{\boldsymbol{r}}_i$$

Notice that it is a function of **r**, because the separation vectors $\mathbf{r}_i$ depend on the location of the field point P.

Example. Find the electric field a distance z above the midpoint between two equal charges q, a distance d from the midpoint.

We calculate the electric field due to the left charge:

$$r = \sqrt{d^2 + z^2}\,, \hat{\boldsymbol{r}} = \frac{d\hat{x} + z\hat{z}}{\sqrt{d^2+z^2}}$$

So $\boldsymbol{E}_{q\,left}(\boldsymbol{r}) = k\frac{q}{\sqrt{d^2+z^2}^2}\frac{d\hat{x}+z\hat{z}}{\sqrt{d^2+z^2}} = k\frac{q}{(d^2+z^2)^{\frac{3}{2}}}(d\hat{\boldsymbol{x}} + z\hat{\boldsymbol{z}})$

Now we calculate the electric field due to the right charge:

$r = \sqrt{d^2 + z^2}\,, \hat{\boldsymbol{r}} = \frac{-d\hat{\boldsymbol{x}}+z\hat{\boldsymbol{z}}}{\sqrt{d^2+z^2}}$

So $\boldsymbol{E}_{q\,right}(\boldsymbol{r}) = k\frac{q}{\sqrt{d^2+z^2}^2}\frac{-d\hat{\boldsymbol{x}}+z\hat{\boldsymbol{z}}}{\sqrt{d^2+z^2}} = k\frac{q}{(d^2+z^2)^{\frac{3}{2}}}(-d\hat{\boldsymbol{x}} + z\hat{\boldsymbol{z}})$

We can add them vectorially:

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_{q\,left}(\boldsymbol{r}) + \boldsymbol{E}_{q\,rigth}(\boldsymbol{r})$$

$$\boldsymbol{E}(\boldsymbol{r}) = k\frac{q}{(d^2+z^2)^{\frac{3}{2}}}(d\hat{\boldsymbol{x}} + z\hat{\boldsymbol{z}}) + k\frac{q}{(d^2+z^2)^{\frac{3}{2}}}(-d\hat{\boldsymbol{x}} + z\hat{\boldsymbol{z}})$$

$$\boldsymbol{E}(\boldsymbol{r}) = k\frac{2qz}{(d^2+z^2)^{\frac{3}{2}}}\hat{\boldsymbol{z}}$$

Continues Charge Distribution

Our definition of the eclectic field, assumes that the source of the field is a collection of discrete point charges q_i. If, instead, the charge is distributed continuously over some region, the sum becomes an integral:

$$\boxed{\boldsymbol{E}(\boldsymbol{r}) = k\int \frac{1}{r^2}\hat{\boldsymbol{r}}\,dq}$$

Where q is spread out over a line, an area, or a volume.

Line	$dq = \lambda\, dl$
Area	$dq = \sigma\, da$
Volume	$dq = \rho\, d\tau$

Example. Find the Electric field anywhere in space, due to an infinite straight line, that carries a uniform line charge λ.

A line segment dl that is at distance r (x along the line and distance z perpendicular to the line) produces an electric field:

$r = \sqrt{x^2 + z^2}, \hat{\boldsymbol{r}} = \frac{-x\hat{\boldsymbol{x}}+z\hat{\boldsymbol{z}}}{\sqrt{x^2+z^2}}$

$$\boldsymbol{E}(\boldsymbol{r}) = k\int_{-\infty}^{\infty}\frac{\lambda}{r^2}\hat{\boldsymbol{r}}\,dx = k\int_{-\infty}^{\infty}\frac{\lambda}{\sqrt{x^2+z^2}^2}\frac{-x\hat{\boldsymbol{x}}+z\hat{\boldsymbol{z}}}{\sqrt{x^2+z^2}}\,dx$$

$$\boldsymbol{E}(\boldsymbol{r}) = k\int_{-\infty}^{\infty}\frac{\lambda}{(x^2+z^2)^{\frac{3}{2}}}(-x\hat{\boldsymbol{x}} + z\hat{\boldsymbol{z}})\,dx$$

$$\boldsymbol{E}(\boldsymbol{r}) = -k\lambda \int_{-\infty}^{\infty} \frac{x}{(x^2+z^2)^{\frac{3}{2}}} \hat{\boldsymbol{x}}\, dx + k\lambda \int_{-\infty}^{\infty} \frac{z}{(x^2+z^2)^{\frac{3}{2}}} \hat{\boldsymbol{z}}\, dx$$

$$\boldsymbol{E}(\boldsymbol{r}) = -k\lambda \int_{-\infty}^{\infty} \frac{x}{(x^2+z^2)^{\frac{3}{2}}} \hat{\boldsymbol{x}}\, dx + k\lambda \int_{-\infty}^{\infty} \frac{z}{(x^2+z^2)^{\frac{3}{2}}} \hat{\boldsymbol{z}}\, dx$$

$$\boldsymbol{E}(\boldsymbol{r}) = -k\lambda \left[-\frac{1}{(x^2+z^2)^{\frac{1}{2}}} \right]_{-\infty}^{\infty} \hat{\boldsymbol{x}} + k\lambda z \left[\frac{x}{z^2\sqrt{x^2+z^2}} \right]_{-\infty}^{\infty} \hat{\boldsymbol{z}}$$

$$\boldsymbol{E}(\boldsymbol{r}) = k\frac{2\lambda}{z} \hat{\boldsymbol{z}} \lim_{x\to\infty} \frac{x}{\sqrt{x^2+z^2}} = k\frac{2\lambda}{z}\hat{\boldsymbol{z}}$$

Example. Find the Electric field anywhere in space, due to an infinite circular plane, that carries a uniform surface charge σ.

A surface segment da is at distance r (x over the surface and distance z perpendicular to the surface) produces an electric field:

$r = \sqrt{x^2+z^2}, \hat{\boldsymbol{r}} = \frac{-x\hat{\boldsymbol{x}}+z\hat{\boldsymbol{z}}}{\sqrt{x^2+z^2}}$

$$\boldsymbol{E}(\boldsymbol{r}) = k\int_S \frac{\sigma}{r^2}\hat{\boldsymbol{r}}\, da$$

$$\boldsymbol{E}(\boldsymbol{r}) = k\int_S \frac{\sigma}{\sqrt{x^2+z^2}^2} \frac{-x\hat{\boldsymbol{x}}+z\hat{\boldsymbol{z}}}{\sqrt{x^2+z^2}}\, da$$

For symmetry reasons the electric field can only point in the z direction. To make the calculations easier, we compute only the component of E in the z direction.

$$E_z = \boldsymbol{E}(\boldsymbol{r}) \cdot \hat{\boldsymbol{z}} = k\int_S \frac{\sigma}{\sqrt{x^2+z^2}^2} \frac{-x\hat{\boldsymbol{x}}+z\hat{\boldsymbol{z}}}{\sqrt{x^2+z^2}}\, da \cdot \hat{\boldsymbol{z}} = k\int_S \frac{\sigma}{\sqrt{x^2+z^2}^2} \frac{z}{\sqrt{x^2+z^2}}\, da$$

We can write a surface element da as a ring at distance x and width dx.

$$E_z = \frac{1}{4\pi\varepsilon_0} \int_0^{\infty} \frac{\sigma z}{(x^2+z^2)^{\frac{3}{2}}} 2\pi x\, dx$$

$$E_z = \frac{\sigma z}{2\varepsilon_0} \int_0^{\infty} \frac{x}{(x^2+z^2)^{\frac{3}{2}}}\, dx = \frac{\sigma z}{2\varepsilon_0} \left[-\frac{1}{(x^2+z^2)^{\frac{1}{2}}} \right]_{x=0}^{\infty} = \frac{\sigma}{2\varepsilon_0}$$

We can summarize the decay for different charge distributions in a table:

Point charge	$\boldsymbol{E}(\boldsymbol{r}) = k\frac{q}{r^2}\hat{\boldsymbol{r}}$	$E(r) \propto \frac{1}{r^2}$
Infinite line	$\boldsymbol{E}(\boldsymbol{r}) = k\frac{2\lambda}{r}\hat{\boldsymbol{r}}$	$E(r) \propto \frac{1}{r}$
Infinite plate	$\boldsymbol{E}(\boldsymbol{r}) = \frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{r}}$	$E(r) = constant$

Dipoles

Assume we have a charge configuration of +q and −q at a distance d apart. This configuration is called a *dipole*.

We can define the dipole moment as:

$$\boxed{p = qd}$$

As a dipole is placed in a constant electric field, the torque $T = \boldsymbol{d} \times \boldsymbol{F} = \boldsymbol{d} \times q\boldsymbol{E} = q\boldsymbol{d} \times \boldsymbol{E} = \boldsymbol{p} \times \boldsymbol{E}$.

$$\boxed{T = \boldsymbol{d} \times \boldsymbol{F} = \boldsymbol{p} \times \boldsymbol{E}}$$

Materials

In a *conductor*, charge is free to move.

In an *insulator*, charge is NOT free to move.

Insulators that contain molecular dipoles are called *dielectrics*.

Flux

The amount of electric field going through a surface is called *flux*. The flux Φ_E of **E** through a surface S is a scalar:

$$\boxed{\Phi_E \equiv \int_S \boldsymbol{E} \cdot d\boldsymbol{a}}$$

Gauss's Law

Assume we have a point charge q, is produces an electric field $E = k\frac{q}{r^2}\hat{\boldsymbol{r}}$. This electric field points always away from charge q (or towards q if q is negative). We calculate the flux through a spherical surface with in the centre charge q.

$$\Phi_E = \oint_S k\frac{q}{r^2}\hat{\boldsymbol{r}} \cdot d\boldsymbol{a}$$

Since the normal vector of the spherical surface area is always parallel to the electric field, which points radially away from q, the integral becomes:

$$\Phi_E = \oint_S^0 k\frac{q}{r^2} da = k\frac{q}{r^2}\oint_S da = k\frac{q}{r^2}4\pi r^2 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}4\pi r^2 = \frac{q}{\varepsilon_0}$$

In fact this results holds for every closed surface, since the same amount of field passes through every closed surface that encloses charge is the same:

Gauss's Law

$$\oint_{\boldsymbol{S}} \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{Q_{enc}}{\varepsilon_0}$$

By the divergence theorem:

$$\int_V (\nabla \cdot \boldsymbol{E}) d\tau = \oint_{\boldsymbol{S}} \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{Q_{enc}}{\varepsilon_0}$$

Rewriting Q_{enc} in terms of the charge density ρ, we have $Q_{enc.} = \int_V \rho \, d\tau$. So gauss's law becomes:

$$\int_V (\nabla \cdot \boldsymbol{E}) d\tau = \int_V \frac{\rho}{\varepsilon_0} d\tau$$

This result holds for any volume, so the integrands must be equal:

Gauss's Law in differential form

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

Applications of Gauss's Law

Suppose we have a spherical surface at radius R and total charge q.
From Gauss's law we know that $EA = \frac{Q_{enc}}{\varepsilon_0}$, here:

$$E4\pi r^2 = \frac{Q_{enc}}{\varepsilon_0}$$

Rewriting the equation:

$$E(r) = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2} = k\frac{Q_{enc}}{r^2}$$

Meaning that the electric field at r from the centre of the sphere equals 0 if r<R, since no charge is enclosed. And the electric field equals $k\frac{Q}{r^2}$ for any r>R. Notice that this is equal to the electric field of a point charge Q.

Consider again an infinite straight line that carries a uniform line charge λ. Imagine a Gaussian surface in the shape of a cylinder around a line segment with length L. From Gauss's Law we know that $EA = \frac{Q_{enc}}{\varepsilon_0}$.
Here the electric field is parallel to the base of the cylinder and perpendicular to the other surface area so Gauss's law becomes:

$$E2\pi rL = \frac{\lambda L}{\varepsilon_0}$$

Rewriting the equation:

$$E(r) = \frac{\lambda}{2\pi r\varepsilon_0} = \frac{2\lambda}{4\pi\varepsilon_0 r} = k\frac{2\lambda}{r}$$

As derived earlier.

Example Consider an infinite plane that carries a uniform surface charge σ. Find its electric field.
We draw a Gaussian box extending equal distances above and below the plane. The sides of the box are parallel to the electric field and the top and bottom of the box are perpendicular to the electric field. All the cross-sections parallel to the plane are perpendicular to the electric field, so Gauss's law becomes:

$$E(A_{top} + A_{bottom}) = \frac{\sigma A_{cross}}{\varepsilon_0}$$

Where:

$$A_{top} = A_{bottom} = A_{cross} = A$$

So:

$$E2A = \frac{\sigma A}{\varepsilon_0}$$

Rewriting the equation:

$$E(r) = 2\frac{\sigma}{\varepsilon_0}$$

As derived earlier.

Electrostatic Potential energy

Imagine two charges q and Q a distance d apart. It is very clear that work had to be done to bring those charges at their location. The amount of work to bring a collection of charges at their position is called *Electrostatic Potential Energy* U [J]. For the configuration as describes the amount of work is:

$$U = \int_{\infty}^{d} -Q\boldsymbol{E}\cdot d\boldsymbol{r} = \int_{d}^{\infty} Q\boldsymbol{E}\cdot d\boldsymbol{r} = \int_{d}^{\infty} k\frac{qQ}{r^2}dr = \left[-k\frac{qQ}{r}\right]_{d}^{\infty} = k\frac{qQ}{d}\ [J]$$

Notice that U is independent from the path (E is conservative).

Electric Potential

The *Electric Potential* [V] is the work per unit charge to bring an imaginary charge from infinity to a position. Notice that the electric potential can be different at different locations:

$$V(r) = \int_{r}^{\infty} \boldsymbol{E}\cdot d\tilde{\boldsymbol{r}}$$

Since **E** is a conservative field we can choose any path to come from infinity to r, so the definition of the electric potential is:

$$\boxed{V(r) \equiv \int_{r}^{\infty} \boldsymbol{E}\cdot d\boldsymbol{l}}$$

In some cases it is necessary to calculate the electric potential with respect to a reverence point (in stead of infinity). Integrating $\boldsymbol{E}\cdot d\boldsymbol{l}$ from the common reference point to **r**, it follows that V also satisfies the super position principle.

The *Potential Difference* between two points **a** and **b** is

$$V(\boldsymbol{b}) - V(\boldsymbol{a}) = \int_{b}^{\infty} \boldsymbol{E}\cdot d\boldsymbol{l} - \int_{a}^{\infty} \boldsymbol{E}\cdot d\boldsymbol{l} = \int_{b}^{a} \boldsymbol{E}\cdot d\boldsymbol{l} = -\int_{a}^{b} \boldsymbol{E}\cdot d\boldsymbol{l}$$

So the potential difference is:

$$\boxed{\Delta V = V(\boldsymbol{b}) - V(\boldsymbol{a}) = -\int_{a}^{b} \boldsymbol{E}\cdot d\boldsymbol{l}}$$

The fundamental theorem for gradients states that

$$V(\boldsymbol{b}) - V(\boldsymbol{a}) = \int_{a}^{b} (\nabla V)\cdot d\boldsymbol{l}$$

So

$$\int_{a}^{b} (\nabla V)\cdot d\boldsymbol{l} = -\int_{a}^{b} \boldsymbol{E}\cdot d\boldsymbol{l}$$

Since this is true for any points **a** and **b**, the integrands must be equal:

$$\boxed{\boldsymbol{E} = -\nabla V}$$

Assume there is a point charge q at the origin. Then the potential is

$$V(d) = \int_{\infty}^{d} -\boldsymbol{E} \cdot d\boldsymbol{r} = \int_{d}^{\infty} \boldsymbol{E} \cdot d\boldsymbol{r} = \int_{d}^{\infty} k\frac{q}{r^2} dr = \left[-k\frac{q}{r}\right]_{d}^{\infty} = k\frac{q}{d}\ [V]$$

The potential due to a point charge with respect to infinity is:

$$\boxed{V_p(\boldsymbol{r}) = k\frac{q}{r}}$$

Or for a collection of charges it is:

$$V_p(\boldsymbol{r}) = k\sum_{i=1}^{n} \frac{q_i}{r_i}$$

In particular, for a volume charge it is:

$$\boxed{V(\boldsymbol{r}) = k\int_V \frac{\rho(\bar{\boldsymbol{r}})}{r} d\tau}$$

Example Find the potential inside and outside a spherical shell of radius R that carries a uniform surface charge. Set the reference point at infinity.

From Gauss's law, the field outside is

$$\boldsymbol{E}(\boldsymbol{r}) = k\frac{q}{r^2}\hat{\boldsymbol{r}}$$

Where q is the total charge on the sphere. The field inside is zero.

Since, the field of a spherical shell is like a point charge for $r > R$, we can say that

$$V(\boldsymbol{r}) = k\frac{q}{r}, for\ r > R$$

For $r < R$, we know that $\boldsymbol{E}(\boldsymbol{r}) = 0$. So there is no potential difference inside the sphere. Inside the sphere is an *equipotential*. So the potential inside the sphere equals the potential of the surface:

$$V(\boldsymbol{r}) = k\frac{q}{R}, for\ r < R$$

Interrelation between charge, E-field and Potential

The volumetric charge distribution ρ, the electric field E and the electric potential V are the three fundamental quantities of electrostatics. We can interrelate these quantities by the following equations:

$\rho \to \boldsymbol{E}$	$\boldsymbol{E}(\boldsymbol{r}) = k\int_V \frac{\rho}{r^2}\hat{\boldsymbol{r}}d\tau$
$\boldsymbol{E} \to \rho$	$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$ *(Gauss's Law in differential form)*
$\boldsymbol{E} \to V$	$V(r) \equiv \int_r^\infty \boldsymbol{E} \cdot d\boldsymbol{l}$
$V \to \boldsymbol{E}$	$\boldsymbol{E} = -\nabla V$
$V \to \rho$	$-\nabla^2 V = \frac{\rho}{\varepsilon_0}$ $(V \to \boldsymbol{E} \to \rho)$
$\rho \to \boldsymbol{V}$	$V(\boldsymbol{r}) = k\int_V \frac{\rho}{r}d\tau$

Conductors

In a conductor charge is free to move. In an electrostatic situation all the charges are at rest, meaning there is no electric field inside a conductor. From Gauss's law follows: If **E** is zero, also ρ is zero. Meaning there is no net charge inside a conductor. If there is any charge on the conductor it cannot be at the inside, so it must be on the surface. Since the **E** field is zero inside the conductor, the conductor is an equipotential. Even the potential difference of two points on the surface is zero; because the integral can be take over any path, including a path through the conductor. The electric field is always perpendicular to equipotential surfaces, so the **E** field is perpendicular to the surface, just outside a conductor.

Example Suppose an uncharged conductor has a cavity inside. Somewhere within the cavity is a charge +q. What will happen inside and on the surface of the conductor?

Imagine a closed Gaussian surface enclosing the cavity. Since we know the **E** field inside the conductor is zero, there must be a net charge inside the Gaussian surface. There is one charge q inside the cavity, so a total charge of –q must be distributed over the surface of the cavity. Then if the conductor as a whole is electrically neutral, there must be a charge +q on its outer surface.

Capacitors

Suppose we have two conductors, and we put charge +Q on one and –Q on the other. Since the potential is constant over a conductor, we can speak unambiguously of the potential difference between them:

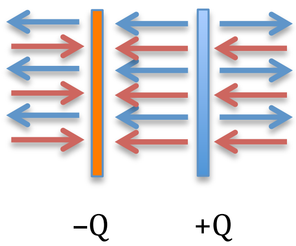
$$V = V_+ - V_- = -\int_{(-)}^{(+)} \boldsymbol{E} \cdot d\boldsymbol{l}$$

The constant of proportionality between the charge Q and the potential difference is called *capacitance* [F].

$$\boxed{C \equiv \frac{Q}{V}}$$

Example Find the capacitance of a parallel-plate capacitor consisting of two metal surfaces of area A held a distance d apart.

Assuming d is small compared to A, we use for a single plate $\boldsymbol{E}(\boldsymbol{r}) = \frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{z}}$.



From the picture it is clear that the **E** field between the plates is two times the field for a single plate: $\boldsymbol{E} = 2\frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$. Everywhere else the field is zero.

The potential difference V between the plates is therefore:

$$V = \frac{\sigma}{\varepsilon_0} d = \frac{Qd}{A\varepsilon_0}$$

So

$$C = \frac{Q}{V} = \frac{Q}{\left(\frac{Qd}{A\varepsilon_0}\right)} = \frac{A\varepsilon_0}{d}$$

In practice the capacitance is linearly proportional to the area A and inverse proportional to the distance d. Usually there is a dielectric material between the plates in order to increase the capacitance. Due to a dielectric the electric field decreases, so the potential difference decreases and the capacitance increases. The capacitance C with the use of a dielectric is proportional to the capacitance without dielectric:

$$\boxed{\begin{gathered} C_r = \kappa C \\ \mathit{Where}\ \kappa\ \mathit{is\ the\ dielectric\ constant} \end{gathered}}$$

For capacitors in Circuits: $C_{Parallel} = C_1 + C_2$ and $\frac{1}{C_{series}} = \frac{1}{C_1} + \frac{1}{C_2}$

Electrostatic Potential Energy for Capacitors

To charge a capacitor in the first place, work has to be done. The work done to bring an infinitesimal piece of charge dq from low to higher potential is:

$$dW = Vdq = \left(\frac{q}{c}\right) dq$$

So the total work necessary, then to go from $q = 0$ to $q = Q$, is:

$$W = \int_0^Q \left(\frac{q}{c}\right) dq = \frac{1}{C}\left[\frac{1}{2}q^2\right]_0^Q = \frac{1}{2}\frac{Q^2}{C}$$

Or, since $Q = CV$

$$\boxed{U = \frac{1}{2}CV^2 \quad \textit{where V is the final potential difference}}$$

Suppose we have a parallel-plate capacitor consisting of two metal surfaces of area A held a distance d apart. Then we know:

$$V = \frac{\sigma}{\varepsilon_0}d = \frac{Qd}{A\varepsilon_0}, C = \frac{Q}{V} = \frac{Q}{\left(\frac{Qd}{A\varepsilon_0}\right)} = \frac{A\varepsilon_0}{d}, and\ E = \frac{Q}{A\varepsilon_0}$$

Substitute both results in the equation for W:

$$W = \frac{1}{2}\frac{A\varepsilon_0}{d}\left(\frac{Qd}{A\varepsilon_0}\right)^2 = \frac{1}{2}\frac{A\varepsilon_0}{d}(Ed)^2 = \frac{1}{2}A\varepsilon_0 E^2 d = \frac{\varepsilon_0}{2}E^2\tau$$

So the energy stored in an electric field **E** is:

$$\boxed{U_E = \frac{\varepsilon_0}{2}\int_V E^2\, d\tau}$$

Rewriting this formula results in the *energy density* of an electric field:

$$\boxed{u_E = \frac{\varepsilon_0}{2}E^2}$$

Electrodynamics

Electric current

Electric current [A] is a net flow of electric charge crossing a given area per unit time:

$$\boxed{I = \frac{dQ}{dt}}$$

The direction of the current corresponds with the flow of positive charges.

The speed at which the charges are moving is called drift velocity. The current is related to the drift velocity by the following equation:

$$I = \frac{dQ}{dt} = nqA\frac{dL}{dt} = nqAv_d$$

Where v_d is the drift velocity.

$$\boxed{v_d = \frac{I}{nqA}}$$

Suppose a wire carrying line charge λ. The current is related to the drift velocity by the following equation:

$$I = \frac{dQ}{dt} = \lambda\frac{dl}{dt} = \lambda v_d$$

Current Density

The current density [Am^{-2}] is the current per unit area. Unlike the current, the current density is a vector quantity.

$$\boxed{\boldsymbol{J} \equiv \frac{dI}{da_\perp}}$$

In general, the current through an area is the flux of the current density over that area:

$$\boxed{I = \int_S \boldsymbol{J} \cdot d\boldsymbol{a}}$$

Since $I = \frac{dQ}{dt} = \frac{\rho A dl}{dt} = \rho A v_d$. We can write $\frac{I}{A} = \rho v_d$ and take the limit $A \to 0$:

$$\boxed{\boldsymbol{J} = \rho \boldsymbol{v}_d}$$

Ohm's Law

Assume a single charge, q in vacuum is accelerated by an electric field **E**. The force on the charge is given by: **F**=q**E**. The velocity of the charge is:

$$\boldsymbol{v} = at = \frac{\boldsymbol{F}}{m_q}t = \frac{q\boldsymbol{E}}{m_{\boldsymbol{q}}}t$$

We know that

$$\boldsymbol{v} = \frac{\boldsymbol{J}}{\rho}$$

So:

$$\frac{q\boldsymbol{E}}{m_{\boldsymbol{q}}}t = \frac{\boldsymbol{J}}{\rho}$$

$$\Leftrightarrow$$

$$\boldsymbol{J} = \frac{\rho q\boldsymbol{E}}{m_{\boldsymbol{q}}}t$$

In materials, due to collisions, the charge is only being accelerated for a very short period of time. The process of accelerating a charge and collision of charges with the material structure, results in a net velocity of the charge. The current density is proportional to the electric field. The proportionality factor σ (not to be confused with surface charge) is called *conductivity*.

Ohm's Law
$\boldsymbol{J} = \sigma\boldsymbol{E}$

Ohm's Law can be represented by an equivalent equation:

$$\boldsymbol{J} = \frac{\boldsymbol{E}}{\rho}$$

Where $\rho = \frac{1}{\sigma}$ is called *resistivity*. (Not to be confused with charge density)

Suppose we have a wire of length L with a constant electric field **E**. Assume **J** is constant and in the direction of **E**.

$$\boldsymbol{J} = \frac{\boldsymbol{E}}{\rho}$$

Then

$$I = \int_S \boldsymbol{J} \cdot d\boldsymbol{a} = I = \int_S \frac{\boldsymbol{E}}{\rho} \cdot d\boldsymbol{a} = \frac{EA}{\rho} = \frac{EL}{\rho}\frac{A}{L} = V\frac{A}{\rho L}$$

Resulting in:

Macroscopic Ohm's Law
$V = IR$

Where $R = \rho\frac{L}{A}$ is called *resistance* [Ω].

Electric Power

In order to move electrons work has to be done. We know that the work per unit charge is called eclectic potential. If we multiply the electric potential by I we have:

$$VI = V\frac{dQ}{dt} = \frac{dW}{dT}$$

We call this expression *Electric Power* [W].

$$\boxed{P = \frac{dW}{dT} = VI}$$

Using Ohm's law, this can be expressed in three equivalent ways:

$$P = VI = I^2R = \frac{V^2}{R}$$

The power in a resistance is dissipated as heat.

Conduction Mechanisms

Conduction occurs in different types of materials. In metallic conductors is current the movement of free electrons. In ionic solution positive and negative ions carry current. Plasmas, which are ionized gases, can carry a current because of the present of ions and electrons.

In *Semiconductors*, current is the movement of both electrons and the lac of electrons in a crystal structure.

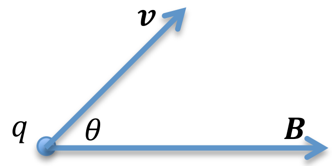
A *Superconductor* offers zero resistance, so electric power can be transmitted without loss. Low temperature is required for a material to be superconducting.

Magnetostatics

Lorentz Force Law

Lorentz force law gives the force [N] on a test charge q, moving with velocity **v** in magnetic field **B**:

$$\boldsymbol{F} = q(\boldsymbol{v}\times\boldsymbol{B})$$



Where $\boldsymbol{F}$ goes into the paper and $F = qvB\sin\theta$.

Cyclotron Motion

The Lorentz force is always perpendicular to the velocity, resulting in a circular motion of the charged particle:

$$qvB = \frac{mv^2}{r} \Leftrightarrow r = \frac{mv}{qB}$$

For the period holds:

$$P = \frac{2\pi r}{v} = \frac{2\pi m}{qB} \Leftrightarrow f = \frac{qB}{2\pi m}$$

Magnetic Forces do no Work

If q moves a distance $\boldsymbol{dl} = \boldsymbol{v}dt$, the work done is:

$$dW = \boldsymbol{F}\cdot\boldsymbol{dl} = q(\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{v}dt = 0$$

This follows because $(\boldsymbol{v}\times\boldsymbol{B})$ is perpendicular to $\boldsymbol{v}$**,** so $(\boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{v} = 0$

Lorentz Force on a Current

Suppose we have a wire carrying a current I. The Lorentz force on a small piece of wire $\boldsymbol{dl}$ is:

$$d\boldsymbol{F} = dq\left(\frac{d\boldsymbol{l}}{dt}\times\boldsymbol{B}\right) = \frac{dq}{dt}(d\boldsymbol{l}\times\boldsymbol{B}) = I(d\boldsymbol{l}\times\boldsymbol{B})$$

So the Lorentz force on a wire is:

$$\boldsymbol{F} = I\int_{wire}(d\boldsymbol{l}\times\boldsymbol{B})$$

Magnetic Field due to Current

The Biot-Savart law gives the magnetic field $\boldsymbol{B}$ due to a steady current I in an element $d\boldsymbol{l}$:

Biot-Savart Law

$$d\boldsymbol{B} = \frac{\mu_0}{4\pi}\frac{I\, d\boldsymbol{l}\times\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}$$

The constant μ_0 is called the *permeability of free space*.

Example. Find the magnetic field anywhere in space, due to an infinite straight wire that carries a current I.

A line segment dl that is at distance r (x along the line and distance s perpendicular to the line) produces a magnetic field:

$$r = \sqrt{x^2 + s^2}$$

$$B = \int_{-\infty}^{\infty} dB = \int_{-\infty}^{\infty} \frac{\mu_0}{4\pi}\frac{I\,dx}{r^2}\frac{s}{r} = \int_{-\infty}^{\infty} \frac{\mu_0 Is}{4\pi}\frac{dx}{(x^2+s^2)^{\frac{3}{2}}} = \frac{\mu_0 Is}{4\pi}\left[\frac{x}{s^2\sqrt{x^2+s^2}}\right]_{-\infty}^{\infty}$$

$$B = \frac{\mu_0 Is}{4\pi}\lim_{p\to\infty}\left[\frac{x}{s^2\sqrt{x^2+s^2}}\right]_{-p}^{p} = \frac{\mu_0 Is}{4\pi}\lim_{p\to\infty}\frac{1}{s^2\sqrt{1+\left(\frac{s}{p}\right)^2}} - \frac{-1}{s^2\sqrt{1+\left(\frac{s}{-p}\right)^2}} = \frac{\mu_0 I}{2\pi s}$$

$$B = \frac{\mu_0 I}{2\pi s}$$

This result can be used to calculate the force between two parallel wires of length L a distance d apart, carrying current I_1 and I_2.

$$B_1 = \frac{\mu_0 I_1}{2\pi s}$$

By the Lorentz force law:

$$\boldsymbol{F}_{12} = I_2\int_0^L (d\boldsymbol{l}\times\boldsymbol{B_1}) = I_2\left(\frac{\mu_0 I_1}{2\pi d}\right)\int_0^L d\boldsymbol{l} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}L$$

Parallel currents attract.
Antiparallel currents repel.

Example. Find the magnetic field a distance s above a ring with radius a carrying a current I.

$$B(s) = \frac{\mu_0 I}{4\pi}\int_{ring}\frac{dl}{r^2}\frac{a}{r} = \frac{\mu_0 I}{4\pi}\int_{ring}\frac{a\,dl}{(a^2+s^2)^{\frac{3}{2}}} = \frac{\mu_0 I}{4\pi}\frac{a}{(a^2+s^2)^{\frac{3}{2}}}2\pi a = \frac{\mu_0 I}{2}\frac{a^2}{(a^2+s^2)^{\frac{3}{2}}}$$

For very large values of s:

$$B(s) = \frac{\mu_0 I}{2}\frac{a^2}{s^3} = \frac{\mu_0 I}{2\pi}\frac{\pi a^2}{s^3} = \frac{\mu_0 I}{2\pi}\frac{A}{s^3} = \frac{\mu_0 \mu}{2\pi s^3}$$

Where $\mu = IA$ is called the magnetic dipole moment. For an N-turn loop $\boldsymbol{\mu} = NI\boldsymbol{A}$.

Magnetic dipoles

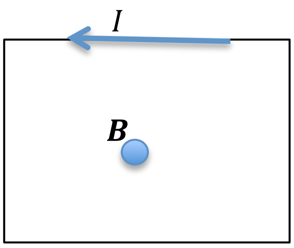
Nobody has even found a magnetic monopole. So the magnetic dipole is the simplest configuration. Magnetic dipoles produce circular magnetic field lines, while a magnetic monopole, if existed, would be the source of a radial magnetic field.

Since magnetic field lines always return we can never have any magnetic flux trough a closed surface:

$$\boxed{\oint_S \boldsymbol{B} \cdot d\boldsymbol{a} = 0}$$

Torque on a current loop

Suppose we have a square loop with sides a carrying a current I which runs clockwise. This loop was placed in a uniform magnetic field $\boldsymbol{B}$ coming out of the paper.



The Lorentz force law tells us there is a force $F_{top} = IaB$ pointing upwards and a force $F_{bottom} = IaB$, so these cancel.

The force on the right side of the loop is $F_{right} = IaB$ pointing to the right. The force on the left side is $F_{left} = IaB$ pointing left.

When we turn this loop around the vertical axis, the forces both sides do not align, resulting in a torque:

$$\boxed{T = \boldsymbol{d} \times \boldsymbol{F} = \boldsymbol{a} \times Ia\boldsymbol{B} = \boldsymbol{\mu} \times \boldsymbol{B}}$$

This torque can be used to make a DC motor. For the DC motor to work, the current has to be reversed to keep the loop spinning.

Magnetism is Matter

All magnetic phenomena are due to electric charges in motion. Electrons orbiting around nuclei are magnetic dipoles. Due to their random orientation most materials are non-magnetic. But in the facility of a magnetic field, these magnetic dipoles tend to align with the magnetic field, and the material has become magnetically polarized or *magnetized*.

Paramagnets acquire a magnetization parallel to the applied magnetic field. These materials exhibit much weaker magnetism.

Diamagnets acquire a magnetic polarization opposite to the applied B-field. Magnets repel these materials.

In *ferromagnetic* materials the magnetic polarization retains after the external field has been removed. These materials become permanent magnets. Most ferromagnets are made from iron.

Ampère's Law

Suppose an infinite wire carrying a current I going into the paper. The magnetic field around this wire is in clockwise direction and has magnitude $B = \frac{\mu_0 I}{2\pi s}$. We calculate the closed integral $\oint_\ell \boldsymbol{B} \cdot d\boldsymbol{r}$ of a circular path of radius s. $\boldsymbol{B}$ and $d\boldsymbol{r}$ are always pointing in the same direction:

$$\oint_\ell \boldsymbol{B} \cdot d\boldsymbol{r} = \oint_\ell \frac{\mu_0 I}{2\pi s} dr = \frac{\mu_0 I}{2\pi s} \oint_\ell dr = \frac{\mu_0 I}{2\pi s} 2\pi s = \mu_0 I$$

Notice the answer is independent of s. In fact every closed loop that encloses the wire would give the same answer. Suppose we have a bundle of wires. Each wire that passes through the loop contributes $\mu_0 I$ to the integral:

Ampère's law

$$\oint_\ell \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I_{enc}$$

If the flow of chare is represented by a current density $\boldsymbol{J}$, the closed loop integral becomes:

$$\oint_\ell \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \int_S \boldsymbol{J} \cdot d\boldsymbol{a}$$

Applying strokes theorem:

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot d\boldsymbol{a} = \oint_\ell \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \int_S \boldsymbol{J} \cdot d\boldsymbol{a}$$

Ampère's law becomes:

Ampère's law in differential form

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

Applications of Ampère's Law

Suppose we have an infinite wire with radius R carrying a current I. We apply Ampère's law. Choose a circular path of radius $r > R$ encircling the wire. From Ampère's law we know that $Bl = \mu_0 I_{enc}$, here:

$$B2\pi r = \mu_0 I_{enc}$$

Rewriting the equation:

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

As derived earlier.

We can also calculate the magnetic field inside the wire. Now we choose $r < R$. Ampère's law becomes:

$$B2\pi r = \mu_0 I \frac{r^2}{R^2}$$

Rewriting the equation:

$$B(r) = \frac{\mu_0 I r}{2\pi R^2}$$

Example. Consider an infinite current sheet carrying a current per unit length $\boldsymbol{K}$. We know that $\boldsymbol{B}$ encircles the current clockwise. Here, by symmetry, $\boldsymbol{B}$ must be straight lines along the sheet parallel to $\boldsymbol{K}$.

We choose our Ampèrian loop to be rectangular, with top and bottom parallel to $\boldsymbol{B}$ and sides perpendicular to the sheet, such that it encircles an enclosed current $\boldsymbol{K}l$ and $\boldsymbol{B} \cdot d\boldsymbol{r}$ only has a contribution to the integral on the top and bottom of the loop.

$$\oint_{\ell} \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I_{enc}$$

Becomes:

$$B2l = \mu_0 \boldsymbol{K} l$$

Rewriting the equation:

$$B = \frac{\mu_0 \boldsymbol{K}}{2}$$

Example. A solenoid is a long, tightly wound coil of wire. For $L \gg D$, the magnetic field inside is approximately uniform and outside is near zero. We calculate the magnetic field inside the solenoid by choosing again a rectangular Ampèrian loop of length l, where the top of the loop is inside the solenoid is parallel to $\boldsymbol{B}$ and the bottom is outside the solenoid.
Since there is no magnetic field outside the solenoid and the sides of the loop are perpendicular to the B-field, neither the top nor sides contribute to the integral of Ampère's law. All we have left is:

$$Bl = \mu_0 I_{enc}$$

Where $I_{enc} = nIl$, where n is the number of turns per unit length.

$$Bl = \mu_0 nIl$$

So the magnetic field in a solenoid of length L having N turns is:

$$\boxed{B = \frac{\mu_0 NI}{L}}$$

Comparison of Electrostatics and Magnetostatics

$\oint_S \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{Q_{enc}}{\varepsilon_0}$	$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$	Gauss's law; Electric monopoles create electric field.
$\oint_\ell \boldsymbol{E} \cdot d\boldsymbol{l} = 0$	$\boldsymbol{\nabla} \times \boldsymbol{E} = 0$	Electric fields are conservative.
$\oint_S \boldsymbol{B} \cdot d\boldsymbol{a} = 0$	$\nabla \cdot \boldsymbol{B} = 0$	Magnetic field lines are closed.
$\oint_\ell \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I_{enc}$	$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$	Ampère's law; Steady currents create magnetic field.

We can summarize the decay for different charge distributions and current distributions in a table:

Distribution	Electric field	Magnetic field	Decay
Dipole	$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{qd}{r^3}\hat{\boldsymbol{r}}$	$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi r^3}(3(\boldsymbol{\mu} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{\mu})$	$\frac{1}{r^3}$
Point charge	$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}$	Not found	$\frac{1}{r^2}$
Infinite line	$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{r}\hat{\boldsymbol{r}}$	$\boldsymbol{B}(r) = \frac{\mu_0 \lambda \boldsymbol{v} \times \hat{\boldsymbol{r}}}{2\pi r}$	$\frac{1}{r}$
Infinite plate	$\boldsymbol{E}(\boldsymbol{r}) = \frac{\sigma}{2\varepsilon_0}\hat{\boldsymbol{r}}$	$\boldsymbol{B}(r) = \frac{\mu_0 \boldsymbol{K} \times \hat{\boldsymbol{r}}}{2} sign(\boldsymbol{r})$	$constant$

Electromagnetic Induction

Faraday's Law

When a magnet approaches a loop of wire a current starts to flow. At the moment the magnet comes to rest the current stops. Faraday had an ingenious inspiration: *A changing magnetic field induces an electric field*:

$$\mathcal{E} = \oint_{\ell} \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_{S} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{a}$$

Rewriting the equation in terms of magnetic flux $\Phi_B = \int_S \boldsymbol{B} \cdot d\boldsymbol{a}$, the equation becomes Faraday's law:

Faraday's law

$$\oint_{\ell} \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi_B}{dt}$$

It is this induced magnetic field that accounts for the *electromotive force*, $\mathcal{E} = \oint_{\ell} \boldsymbol{E} \cdot d\boldsymbol{l}$. Indeed the emf is equal to the rate of change of magnetic flux.

We can convert Faraday's law into differential form by applying strokes theorem:

$$\int_{S} (\boldsymbol{\nabla} \times \boldsymbol{E}) \cdot d\boldsymbol{a} = \oint_{\ell} \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_{S} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{a}$$

Faraday's law becomes:

Faraday's law in differential form

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Lenz's Law

The direction of the induced emf and current is described by the minus sign in Faraday's law, but it's easier to get the direction from the law of conservation of energy.

Lenz's law tells that the direction of the induced current must be such that it opposes the change of magnetic flux. In other words, an induced current cannot support a rise in magnetic flux, because then it would blow up.

Applications of Faraday's Law

Suppose a loop of wire of radius a and resistance R. The loop is in a changing magnetic field $\boldsymbol{B}$. The rate of change is $\dot{B} = \frac{d\boldsymbol{B}}{dt}$. Then the induced emf is:

$$\mathcal{E} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{a} = -\frac{dB}{dt}\int_S da = -\pi a^2 \dot{B}$$

So the induced current $I = \frac{\mathcal{E}}{R} = \frac{-\pi a^2}{R}\dot{B}$

Example. Consider a bar sliding on a conducting rails with speed v, increasing the circuit area. Calculate the induced current.

$$\frac{dA}{dt} = \frac{dhx}{dt} = \frac{h\,dx}{dt} = hv$$

So the induced emf is:

$$\mathcal{E} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{a} = -B\frac{dA}{dt} = -Bhv$$

So the induced current is:

$$I = \frac{\mathcal{E}}{R_{circuit}} = \frac{-Bhv}{R_{circuit}}$$

Inductance

Mutual inductance occurs when a changing current in one circuit results in, via changing magnetic flux, in an induced current in an adjacent circuit.

Self-inductance occurs when a changing current in a circuit results in an induced emf that opposes the change in the circuit itself.

The self-inductance [H] of a circuit is defined as the ratio of the magnetic flux through the circuit to the current in the circuit:

$$\boxed{\Phi_B = LI}$$

Consider a long solenoid of cross-sectional area A, length l and a total of N turns. The magnetic field in the solenoid is:

$$B = \frac{\mu_0 NI}{l}$$

Since the magnetic field is perpendicular to the cross-section the magnetic flux is:

$$\Phi_B = B\int_S d\boldsymbol{a} = \frac{\mu_0 NI}{l}NA = \frac{\mu_0 N^2 IA}{l}$$

So the self-inductance of the solenoid is:

$$\boxed{L = \mu_0 N^2 \frac{A}{l}}$$

LRC-circuits

The self-inductance L is:

$$\Phi_B = LI$$

And by Faradays law:

$$\mathcal{E}_{ind} = -\frac{d\Phi_B}{dt} = -\frac{dLI}{dt} = -L\frac{dI}{dt}$$

So:

$$\boxed{\mathcal{E}_{ind} = -L\frac{dI}{dt}}$$

Such a differential equation can be derived for the capacitance as well:

$$C \equiv \frac{Q}{V}$$

Becomes:

$$V = \frac{Q}{C}$$

Taking the derivative with respect to time:

$$\frac{dV}{dt} = \frac{1}{C}\frac{dQ}{dt} = \frac{I}{C}$$

So:

$$\boxed{I = C\frac{dV}{dt}}$$

Example. Suppose an RL-circuit. Calculate the current I as a function of time. Take for $t = 0$ the moment you connect the circuit to a battery of V volts.

Apply Faraday's Law:

$$\mathcal{E} = IR - V = -L\frac{dI}{dt}$$

The solution of the differential equation is:

$$I = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

Energy in Magnetic Fields

It takes energy to start a current flowing in a circuit. This is not the power delivered by the battery in the steady situation. This is the fixed amount of you get back when the battery is removed from the circuit.

That energy stored in the solenoids magnetic field is:

$$W = \int_{time} P\, dt = \int_{time} -\mathcal{E} I\, dt = \int_{time} L \frac{dI}{dt} I\, dt = L \int_{time} I\, dI = \frac{1}{2} L I^2$$

So:

$$\boxed{U_B = \frac{1}{2} L I^2}$$

Since this energy was stored inside the solenoids magnetic field. Where $B = \frac{\mu_0 N I}{l}$, so:

$$I = \frac{Bl}{\mu_0 N}$$

And the self-inductance $L = \mu_0 N^2 \frac{A}{l}$.

The stored energy becomes:

$$U_B = \frac{1}{2} L I^2 = \frac{1}{2} \mu_0 N^2 \frac{A}{l} \left(\frac{Bl}{\mu_0 N} \right)^2 = \frac{B^2}{2\mu_0} AL$$

So the energy stored in an electric field **B** is:

$$\boxed{U_B = \frac{1}{2\mu_0} \int_V B^2\, d\tau}$$

Rewriting this formula results in the *energy density* of a magnetic field:

$$\boxed{u_E = \frac{B^2}{2\mu_0}}$$

Paradox in Ampère's Law

Suppose a capacitor is being charged by a current I. The electric field between the capacitor plates is $E = \frac{\sigma}{\varepsilon_0}$, so:

$$\frac{dE}{dt} = \frac{1}{\varepsilon_0}\frac{d\sigma}{dt} = \frac{1}{A\varepsilon_0}\frac{Ad\sigma}{dt} = \frac{I}{A\varepsilon_0}$$

The magnetic field created by the charging wire can be calculated by applying Ampère's law:

$$\oint_\ell \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0 I_{enc}$$

Becomes:

$$B2\pi r = \mu_0 I_{enc}$$

So the magnetic field is:

$$B = \frac{\mu_0 I_{enc}}{2\pi r}$$

When we want to find I_{enc} it depends now on the open surface we define attached to me Ampèrian loop; when we take a flat surface $I_{enc} = I$, but when we choose an open surface between the plates of the capacitor then $I_{enc} = 0$.

Maxwell came with a solution for this paradox: The changing electric field $\frac{dE}{dt}$ can also induce a magnetic field:

$$\frac{dE}{dt} = \frac{I}{A\varepsilon_0}$$

So:

$$I = A\varepsilon_0\frac{dE}{dt} = \varepsilon_0\frac{d\Phi_E}{dt}$$

Using this insight Ampère's law becomes:

Modified Ampère's law

$$\oint_\ell \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0 I_{enc} + \mu_0\varepsilon_0\frac{d\Phi_E}{dt}$$

Where $\varepsilon_0\frac{d\Phi_E}{dt}$ is called the *displacement current*.

Applying strokes theorem:

$$\int_S (\boldsymbol{\nabla}\times\boldsymbol{B})\cdot d\boldsymbol{a} = \oint_\ell \boldsymbol{B}\cdot d\boldsymbol{r} = \mu_0\int_S \boldsymbol{J}\cdot d\boldsymbol{a} + \mu_0\varepsilon_0\frac{d}{dt}\int_S \boldsymbol{E}\cdot d\boldsymbol{a}$$

Ampère's law becomes:

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\varepsilon_0\frac{d\boldsymbol{E}}{dt}$$

Maxwell's equations

The four complete laws of electromagnetism are collectively called Maxwell's equations:

Gauss, E	$\oint_S \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{Q_{enc}}{\varepsilon_0}$	$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$	Electric field lines begin and end on charges.
Gauss, B	$\oint_S \boldsymbol{B} \cdot d\boldsymbol{a} = 0$	$\nabla \cdot \boldsymbol{B} = 0$	Magnetic field lines don't begin or end.
Faraday	$\oint_\ell \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi_B}{dt}$	$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$	Changing magnetic flux produces electric field.
Modified Ampère	$\oint_\ell \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$	$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{d\boldsymbol{E}}{dt}$	Electric current and changing electric flux produce magnetic field.

Electromagnetic Waves

In vacuum there are no electric charges, so the Maxwell equations become:

$\oint_S \boldsymbol{E} \cdot d\boldsymbol{a} = 0$	$\oint_S \boldsymbol{B} \cdot d\boldsymbol{a} = 0$
$\oint_\ell \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d\Phi_B}{dt}$	$\oint_\ell \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$

Combining the last two equations leads to:

$$\frac{\partial B}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \Leftrightarrow \frac{\partial^2 B}{\partial x^2} = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial x \partial t} \Leftrightarrow \frac{\partial^2 B}{\partial t \partial x} = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

And

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \Leftrightarrow \frac{\partial^2 E}{\partial x^2} = -\frac{\partial^2 B}{\partial x \partial t} \Leftrightarrow \frac{\partial^2 E}{\partial t \partial x} = -\frac{\partial^2 B}{\partial t^2}$$

Now we can conclude that:

$$\frac{\partial^2 E}{\partial x^2} = -\frac{\partial^2 B}{\partial x \partial t} = \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}$$

And

$$\frac{\partial^2 B}{\partial x^2} = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial x \partial t} = \varepsilon_0 \mu_0 \frac{\partial^2 B}{\partial t^2}$$

So both E and B are satisfying the wave equation:

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y(x,t)}{\partial t^2}$$

Where $v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3.0 \cdot 10^8 = c$

Solving both equations with respect to Maxwell's equations give:

$$E(x,t) = E_p \sin(kx - \omega t)$$

$$B(x,t) = B_p \sin(kx - \omega t)$$

$$\frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = c$$

$$E = cB$$

The constant c is called the speed of light. And light is an electromagnetic wave of any frequency or wavelength. These electromagnetic waves come in a vast range of frequencies and wavelengths, from radio waves to gamma rays. The human eye can see electromagnetic waves with wavelengths of about 380nm to 780nm as *visible light*.

Poynting vector

Electromagnetic wave containing electric and magnetic fields carry energy. The energy density of electric fields is:

$$u_E = \frac{1}{2}\varepsilon_0 E^2$$

And je energy density of magnetic fields is:

$$u_B = \frac{1}{2\mu_0} B^2$$

Since for electromagnetic waves $B = \frac{E}{c}$ and $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ the equation becomes:

$$u_B = \frac{1}{2\mu_0}\left(\frac{E}{c}\right)^2 = \frac{1}{2\mu_0} E^2 \varepsilon_0 \mu_0 = \frac{1}{2}\varepsilon_0 E^2 = u_E$$

So the total energy density in an electromagnetic wave is:

$$u = u_E + u_B = 2u_E = \varepsilon_0 E^2 = \varepsilon_0 EBc$$

The energy passing through an area per second is (**power per square meter**):

$$\frac{dU}{dt} = uAc = \varepsilon_0 EBc^2 A = \frac{EB}{\mu_0} A$$

We define the *Poynting vector* $S = \frac{1}{A}\frac{dU}{dt}$, so S becomes:

$$\boxed{S \equiv \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}}$$

The average of S over time is:

$$\bar{S} = \frac{1}{P}\int_{Peroid} \frac{E_p B_p}{\mu_0} \sin^2\theta \, dt = \frac{E_p B_p}{2\mu_0}$$

The *Average intensity* is:

$$\boxed{\bar{S} = \frac{E_p B_p}{2\mu_0} = \frac{c B_p^2}{2\mu_0} = \frac{E_p^2}{2c\mu_0}}$$

Radiation pressure

Besides energy, electromagnetic waves also carry momentum P.

$$P = mv$$

Since $F = m\frac{dv}{dt}$, it holds that:

$$F = \frac{dp}{dt}$$

The energy:

$$dU = Fdr = \frac{dp}{dt} dr = dp\, c$$

So the momentum of an electromagnetic wave is:

$$\boxed{p = U/c}$$

The average momentum per unit area is:

$$\frac{p}{A} = \frac{U}{Ac} = \frac{\bar{S}}{c}$$

We can also call this *radiation pressure*, because pressure is:

$$P = \frac{F}{A} = \frac{dp}{A\, dt} = \frac{1}{A} \frac{dU}{C\, dt} = \frac{S}{c}$$

Polarization

In electromagnetic waves the E-field and the B-filed are perpendicular to each other and both are perpendicular to the direction of propagation of the wave. But any orientation is allowed.

The direction of the electric field with respect to the direction of propagation defines the direction of the wave's *polarization*.

Typical sources of light emit a max of polarizations:
The sun emits a combination of waves with uniformly distribution of polarization. Sunlight is *unpolarized*.
Partially polarized light is a combination of waves with randomly distributed polarizations centred around one direction.
If the oscillation of the electric fields is in a single direction the waves are *linear polarized* and when the direction rotates at the optical frequency the wave are *circular* or *elliptical polarized*.

In some materials only one polarization passes the material. Only the polarization along the *transmission axis* can pass. Crossing two polarizers results in no transmission.

When a wave passes a polarizer it emerges with a reduced intensity given by Malus's law:

$$\boxed{S = S_0 \cos^2 \theta}$$

When unpolarized light passes such a polarizer. The intensity will be:

$$S = \frac{S_0}{2}$$

Sources of Electromagnetic waves

Electromagnetic waves are generated ultimately by accelerated eclectic charge:
Radio waves of about are generated by alternating currents in metal antennas. Molecular vibrations and rotation produce *infrared waves*. Visible light arises largely from atomic-scale processes and *X-rays* are produces in the rapid deceleration of electric charge. Lastly *Gamma rays* result from nuclear processes.