Mathematics 3, toets 3. AESB1310, 14 april 2014.

1. parametrisering C: $\underline{r}(t) = \begin{pmatrix} t \\ \ln t \end{pmatrix}$ $1 \le t \le 3$

$\underline{r}'(t) = \begin{pmatrix} 1 \\ 1/t \end{pmatrix}$, $|\underline{r}'(t)| = \sqrt{1 + \left(\frac{1}{t}\right)^2}$

$$\int_C x^2 ds = \int_1^3 t^2 |\underline{r}'(t)| dt = \int_1^3 t^2 \sqrt{1+\left(\tfrac{1}{t}\right)^2}\, dt =$$

$$\int_1^3 t\sqrt{t^2+1}\, dt = \left[\tfrac{1}{2}\tfrac{2}{3}(t^2+1)^{3/2}\right]_1^3 = \tfrac{1}{3}\left(10\sqrt{10} - 2\sqrt{2}\right).$$

2a) $P = 2xy^2+3$, $Q = 2x^2y + 3y$.

$\frac{\partial P}{\partial y} = 4xy$, $\frac{\partial Q}{\partial x} = 4xy$, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, dus $\underline{G}$ is conservatief.

Potentiaalfunctie:

$$\begin{cases} \frac{\partial f}{\partial x} = 2xy^2+3 \\ \frac{\partial f}{\partial y} = 2x^2y+3y. \end{cases} \longrightarrow f = x^2y^2 + 3x + g(y), \rightarrow \frac{\partial f}{\partial y} = 2x^2y + g'(y).$$

Dus $g'(y) = 3y$, $g(y) = \frac{3}{2}y^2 + C$

$f = x^2y^2 + 3x + \frac{3}{2}y^2$ is dus een potentiaalfunctie

b) $$\int_K \underline{G}\cdot d\underline{r} = \int_K \nabla f \cdot d\underline{r} = f(\underline{r}(b)) - f(\underline{r}(a))$$

$$= f(-1,0) - f(1,0) = -3 - 3 = -6$$

3. $\underline{r}_u \times \underline{r}_v = \begin{pmatrix} v \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} u \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -v-u \\ v-u \end{pmatrix}$

$$|\underline{r}_u \times \underline{r}_v| = \sqrt{4 + (-v-u)^2 + (v-u)^2}$$
$$= \sqrt{4 + 2u^2 + 2v^2}$$

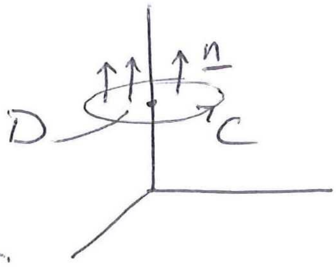
$$A(S) = \iint_S dS = \iint_H |\underline{r}_u \times \underline{r}_v| \, du\, dv =$$

(met $H = \{(u,v) \mid u^2 + v^2 \leq 6)$

$$\iint_H \sqrt{4 + 2u^2 + 2v^2}\, du\, dv = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{6}} \sqrt{4 + 2r^2}\, r\, dr\, d\theta =$$

$$2\pi \left[\frac{1}{6} (4 + 2r^2)^{3/2} \right]_0^{\sqrt{6}} = \frac{\pi}{3}(64 - 8) = \frac{56}{3}\pi$$

4. a) $\text{curl}\, \underline{F} = \begin{pmatrix} 1 \\ -y \\ 1+z \end{pmatrix}$, $\text{div}\, \underline{F} = 0$



b) $\iint_D \text{curl}\, \underline{F} \cdot d\underline{S} = \iint_D \text{curl}\, \underline{F} \cdot \underline{n}\, dS$

$$= \iint_D \begin{pmatrix} \cdots \\ \cdots \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dS = 4\, A(D) = 16\pi$$

op D: $1 + z = 1 + 3 = 4$.

c) $\int_C \underline{F} \cdot d\underline{r} = \iint_D \text{curl}\, \underline{F} \cdot d\underline{S} = 16\pi$ (Stokes)

Merk op dat de oriëntering van C en D met elkaar kloppen via kurkentrekker regel.

4. d) Bol B is een gesloten oppervlak en V het gebied binnen deze bol.

Volgens de divergentiestelling geldt nu:

$$\iint_B \underline{F} \cdot d\underline{S} = \iiint_V \operatorname{div} \underline{F}\, dV = 0 \quad (\text{want } \operatorname{div} \underline{F} = 0)$$

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4 b) alternatief 1. Parametriseer D: $\underline{r}(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 3 \end{pmatrix}$

$H: \ 0 \leq \theta \leq 2\pi, \ 0 \leq r \leq 2$

$$\underline{r}_r \times \underline{r}_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$ orientering juist.

$$\iint_D \operatorname{curl} \underline{F} \cdot d\underline{S} = \iint_H \operatorname{curl} \underline{F} \cdot (\underline{r}_r \times \underline{r}_\theta)\, dr\, d\theta =$$

$$\int_0^{2\pi}\int_0^2 \begin{pmatrix} 1 \\ -r\cos\theta \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} dr\, d\theta = 2\pi \int_0^2 4r\, dr = 16\pi$$

alternatief 2. Parametriseer D: $\underline{r}(x,y) = \begin{pmatrix} x \\ y \\ 3 \end{pmatrix}$,

$H: \ x^2 + y^2 \leq 4.$

$$\underline{r}_x \times \underline{r}_y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$ orientering juist.

$$\iint_D \operatorname{curl} \underline{F} \cdot d\underline{S} = \iint_H \begin{pmatrix} -- \\ -- \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dA = 4\pi \cdot 2^2 = 16\pi.$$

alternatief 3. Bereken $\int_C \underline{F} \cdot d\underline{r}$ (4 c) en gebruik Stokes.

4c) alternatief.

Parametriseer C: $r(t) = \begin{pmatrix} 2\cos t \\ 2\sin t \\ 3 \end{pmatrix}$ $\quad 0 \le t \le 2\pi$.

$\underline{r}'(t) = \begin{pmatrix} -2\sin t \\ 2\cos t \\ 0 \end{pmatrix}$.

$$\int_C \underline{F} \cdot d\underline{r} = \int_0^{2\pi} \begin{pmatrix} -6\sin t \\ 2\cos t + 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ 2\cos t \\ 0 \end{pmatrix} dt =$$

$$\int_0^{2\pi} (12\sin^2 t + 4\cos^2 t + 6\cos t)\, dt =$$

$$\int_0^{2\pi} (4 + 8\sin^2 t + 6\cos t)\, dt =$$

$$\int_0^{2\pi} (4 + (4 - 4\cos 2t) + 6\cos t)\, dt =$$

$$\Big[8t - 2\sin 2t + 6\sin t \Big]_0^{2\pi} = 16\pi.$$