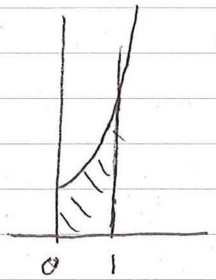
Toets 2 Mathematics 3, AESB1310, 28-03-2014.

1. a)



$$A(D) = \int_0^1 e^x\,dx = \left[e^x\right]_0^1 = e-1.$$

b) formules voor het massamiddelpunt $(\bar{x}, \bar{y})$.

$$\bar{x} = \frac{1}{m}\iint_D x\rho\,dA \quad , \quad \bar{y} = \frac{1}{m}\iint_D y\rho\,dA.$$

$$m = \iint_D \rho\,dA = \rho\iint_D dA = \rho\,A(D) = \rho(e-1).$$

$$\bar{x} = \frac{\rho}{m}\int_0^1\int_0^{e^x} x\,dy\,dx = \frac{\rho}{\rho(e-1)}\int_0^1 xe^x\,dx =$$

$$\frac{1}{e-1}\left[xe^x - e^x\right]_0^1 = \frac{1}{e-1}.$$

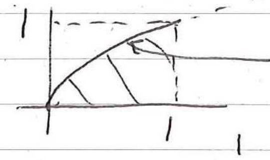
$$\bar{y} = \frac{\rho}{m}\int_0^1\int_0^{e^x} y\,dy\,dx = \frac{1}{e-1}\int_0^1\left[\tfrac{1}{2}y^2\right]_0^{e^x}dx =$$

$$\frac{1}{e-1}\int_0^1 \frac{1}{2}e^{2x}\,dx = \frac{1}{e-1}\left[\frac{1}{4}e^{2x}\right]_0^1 = \frac{1}{4}\,\frac{e^2-1}{e-1}$$

$$= \frac{1}{4}(e+1)$$

2. $y^2 \le x \le 1$

$0 \le y \le 1$



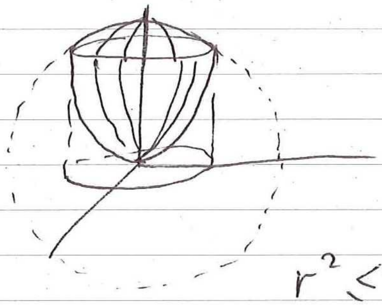
$x = y^2 \quad y = \sqrt{x}$. $\quad 0 \le y \le \sqrt{x}$

$0 \le x \le 1$

$$\int_0^1\int_{y^2}^1 \frac{2y}{x^2+1}\,dx\,dy = \int_0^1\int_0^{\sqrt{x}} \frac{2y}{x^2+1}\,dy\,dx = \int_0^1 \frac{1}{x^2+1}\left[y^2\right]_0^{\sqrt{x}}dx$$

$$= \int_0^1 \frac{x}{x^2+1}\,dx = \left[\frac{1}{2}\ln(x^2+1)\right]_0^1 = \frac{1}{2}\ln 2$$

3.



paraboloïde: $z = x^2+y^2 \quad z = r^2$

bol: $x^2+y^2+z^2 = 6 \quad r^2+z^2 = 6$

$$z = \sqrt{6-r^2}$$

$$r^2 \leq z \leq \sqrt{6-r^2},$$

grenzen van r;
r is maximaal op de doorsnede van de paraboloïde en de bol.

$z = r^2$ en $z^2+r^2 = 6$ dus $r^4+r^2 = 6$

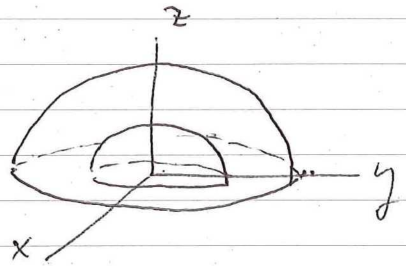
$$r^4+r^2-6 = 0$$
$$(r^2+3)(r^2-2) = 0$$
$$r^2 = 2$$
$$r = \sqrt{2}.$$

$$0 \leq r \leq \sqrt{2}$$

grenzen θ: $\quad 0 \leq \theta \leq 2\pi.$

$$\iiint_H f(x,y,z)\,dV = \int_{\theta=0}^{2\pi}\int_{r=0}^{\sqrt{2}}\int_{z=r^2}^{\sqrt{6-r^2}} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

4.



$$\begin{cases} 1 \leq \rho \leq 2 \\ 0 \leq \varphi \leq \frac{\pi}{2} \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$\iiint_E \frac{z}{\sqrt{x^2+y^2+z^2}}\,dV =$$

$$\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\frac{\pi}{2}}\int_{\rho=1}^{2} \frac{\rho\cos\varphi}{\rho}\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta =$$

$$2\pi\int_0^{\frac{\pi}{2}}\cos\varphi\sin\varphi\left[\tfrac{1}{3}\rho^3\right]_1^2 d\varphi =$$

$$\pi\left[\sin^2\varphi\right]_0^{\frac{\pi}{2}}\cdot\frac{7}{3} = \frac{7}{3}\pi.$$

5. parametriseer C: $\underline{r}(t) = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + t\left\{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\right\}$

$$\underline{r}(t) = \begin{pmatrix} 1-t \\ 1+t \\ -2+3t \end{pmatrix} \qquad 0 \le t \le 1$$

$$\underline{r}'(t) = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \qquad \underline{F}(\underline{r}(t)) = \begin{pmatrix} 1-t^2 \\ 3t \\ 3-t \end{pmatrix}.$$

$$\int_C \underline{F} \cdot d\underline{r} = \int_0^1 \underline{F} \cdot \underline{r}'(t)\, dt =$$

$$\int_0^1 \begin{pmatrix} 1-t^2 \\ 3t \\ 3-t \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} dt = \int_0^1 \left(-(1-t^2) + 3t + 3(3-t)\right) dt$$

$$\int_0^1 (t^2+8)\, dt = \left[\tfrac{1}{3} t^3 + 8t\right]_0^1 = 8\tfrac{1}{3} = \frac{25}{3}$$

2. alternatief (zonder verandering integratievolgorde)

$$\int_0^1 \int_{y^2}^1 \frac{2y}{x^2+1}\,dx\,dy = \int_0^1 2y\left[\arctan x\right]_{y^2}^1 dy =$$

$$\int_0^1 2y\left(\arctan 1 - \arctan y^2\right)dy =$$

$$\left[\frac{\pi}{4}y^2\right]_0^1 - \int_0^1 2y \arctan y^2\,dy \underset{y^2=t}{=} \frac{\pi}{4} - \int_0^1 \arctan t\,dt$$

$$= \frac{\pi}{4} - \left[t\arctan t\right]_0^1 + \int_0^1 \frac{t}{1+t^2}\,dt =$$

$$\frac{\pi}{4} - \frac{\pi}{4} + \frac{1}{2}\left[\ln\left(1+t^2\right)\right]_0^1 = \frac{1}{2}\ln 2.$$

4. alternatief (met cilinder coordinaten).

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=\sqrt{1-r^2}}^{\sqrt{4-r^2}} \frac{zr}{\sqrt{r^2+z^2}}\,dz\,dr \;+$$

$$\int_{\theta=0}^{2\pi}\int_{r=1}^{2}\int_{z=0}^{\sqrt{4-r^2}} \frac{zr}{\sqrt{r^2+z^2}}\,dz\,dr =$$

$$2\pi\int_{r=0}^{1} r\left[\sqrt{r^2+z^2}\right]_{\sqrt{1-r^2}}^{\sqrt{4-r^2}} dr + 2\pi\int_1^2 r\left[\sqrt{r^2+z^2}\right]_0^{\sqrt{4-r^2}} dr =$$

$$2\pi\int_0^1 r(2-1)\,dr + 2\pi\int_1^2 \left(2r - r^2\right)dr = \ldots = \frac{7}{3}\pi$$